MACMILLAN'S CANADIAN SCHOOL SERIES

ALGEBRA FOR MATRICULATION

(SENIOR LEAVING)

HALL & KNIGHT.

Birds of Ontario in relation to Agriculture

Insects Injurious to Vegetation

+ Fungus Diseases affecting vegetation

Department of Agriculture

Toronto

Ont.

168 P

$$x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)\{(x-y)^2+(y-z)^2+(z-x)^2\}=$$

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ALGEBRA FOR MATRICULATION

(SENIOR LEAVING)

BEING THE FIRST XIX CHAPTERS

OF

HIGHER ALGEBRA

BY

H. S. HALL, M.A.

AND

S. R. KNIGHT, B.A.

TORONTO

THE MACMILLAN COMPANY OF CANADA

LIMITED

1908

$\Sigma n = \frac{n(n+1)}{2}$

$\Sigma 2n = n(n+1)$

$\Sigma 2n-1 = n^2$

$\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$

$\Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

$i = \sqrt{-1}$

First Edition 1906
Reprinted 1908

111 2 chapters on Ratio Proport

2. Important principles—

1. If the product be constant, the sum is least when they a[re] equal.

Ex — Let product. = 16. (8×2) Sum=

" " = 16. (4×4) "

PREFACE.

THE present volume, containing the first nineteen chapters of Hall and Knight's *Higher Algebra*, is issued separately for the use of Canadian students. It may, perhaps, be found useful for others whose study of Algebra does not extend beyond the range of these chapters.

H. S. HALL.

Aug. 1906.

(2) If the sum of 2 no be constan[t] their product is greatest whe[n] they are equal.

Let sum = 10. (8+2) = P = 16.

" " = 10. (5+5) = P = 25.

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HIGHER ALGEBRA.

CHAPTER I.

RATIO.

1. Definition. Ratio is the relation which one quantity bears to another of the *same* kind, the comparison being made by considering what multiple, part, or parts, one quantity is of the other.

The ratio of A to B is usually written $A : B$. The quantities A and B are called the *terms* of the ratio. The first term is called the antecedent, the second term the consequent.

2. To find what multiple or part A is of B, we divide A by B; hence the ratio $A : B$ may be measured by the fraction $\frac{A}{B}$, and we shall usually find it convenient to adopt this notation.

In order to compare two quantities they must be expressed in terms of the same unit. Thus the ratio of £2 to 15*s.* is measured by the fraction $\frac{2 \times 20}{15}$ or $\frac{8}{3}$.

Note. A ratio expresses the *number* of times that one quantity contains another, and therefore *every ratio is an abstract quantity.*

3. Since by the laws of fractions,

$$\frac{a}{b} = \frac{ma}{mb},$$

it follows that the ratio $a : b$ is equal to the ratio $ma : mb$; that is, *the value of a ratio remains unaltered if the antecedent and the consequent are multiplied or divided by the same quantity.*

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4. Two or more ratios may be compared by reducing their equivalent fractions to a common denominator. Thus suppose $a : b$ and $x : y$ are two ratios. Now $\frac{a}{b} = \frac{ay}{by}$, and $\frac{x}{y} = \frac{bx}{by}$; hence the ratio $a : b$ is greater than, equal to, or less than the ratio $x : y$ according as ay is greater than, equal to, or less than bx.

5. The ratio of two fractions can be expressed as a ratio of two integers. Thus the ratio $\frac{a}{b} : \frac{c}{d}$ is measured by the fraction $\dfrac{\frac{a}{b}}{\frac{c}{d}}$, or $\frac{ad}{bc}$; and is therefore equivalent to the ratio $ad : bc$.

6. If either, or both, of the terms of a ratio be a surd quantity, then no two integers can be found which will *exactly* measure their ratio. Thus the ratio $\sqrt{2} : 1$ cannot be exactly expressed by any two integers.

7. Definition. If the ratio of any two quantities can be expressed exactly by the ratio of two integers, the quantities are said to be **commensurable**; otherwise, they are said to be **incommensurable.**

Although we cannot find two integers which will exactly measure the ratio of two incommensurable quantities, we can always find two integers whose ratio differs from that required by as small a quantity as we please.

Thus $$\frac{\sqrt{5}}{4} = \frac{2 \cdot 236067\ldots}{4} = \cdot 559016\ldots$$

and therefore $$\frac{\sqrt{5}}{4} > \frac{559016}{1000000} \text{ and } < \frac{559017}{1000000};$$

so that the difference between the ratios $559016 : 1000000$ and $\sqrt{5} : 4$ is less than $\cdot 000001$. By carrying the decimals further, a closer approximation may be arrived at.

8. Definition. Ratios are *compounded* by multiplying together the fractions which denote them; or by multiplying together the antecedents for a new antecedent, and the consequents for a new consequent.

Example. Find the ratio compounded of the three ratios

$$2a : 3b,\ 6ab : 5c^2,\ c : a$$

$$\text{The required ratio} = \frac{2a}{3b} \times \frac{6ab}{5c^2} \times \frac{c}{a}$$
$$= \frac{4a}{5c}.$$

9. DEFINITION. When the ratio $a : b$ is compounded with itself the resulting ratio is $a^2 : b^2$, and is called the **duplicate ratio** of $a : b$. Similarly $a^3 : b^3$ is called the **triplicate ratio** of $a : b$. Also $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the **subduplicate ratio** of $a : b$.

Examples. (1) The duplicate ratio of $2a : 3b$ is $4a^2 : 9b^2$.
(2) The subduplicate ratio of $49 : 25$ is $7 : 5$.
(3) The triplicate ratio of $2x : 1$ is $8x^3 : 1$.

10. DEFINITION. A ratio is said to be a ratio of *greater inequality*, of *less inequality*, or of *equality*, according as the antecedent is *greater than*, *less than*, or *equal to* the consequent.

11. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both its terms.*

Let $\frac{a}{b}$ be the ratio, and let $\frac{a+x}{b+x}$ be the new ratio formed by adding x to both its terms.

Now
$$\frac{a}{b} - \frac{a+x}{b+x} = \frac{ax - bx}{b(b+x)}$$
$$= \frac{x(a-b)}{b(b+x)};$$

and $a - b$ is positive or negative according as a is greater or less than b.

Hence if $a > b$, $\qquad \frac{a}{b} > \frac{a+x}{b+x};$

and if $\qquad a < b$, $\qquad \frac{a}{b} < \frac{a+x}{b+x};$

which proves the proposition.

Similarly it can be proved that *a ratio of greater inequality is increased, and a ratio of less inequality is diminished, by taking the same quantity from both its terms.*

12. When two or more ratios are equal many useful propositions may be proved by introducing a single symbol to denote each of the equal ratios.

The proof of the following important theorem will illustrate the method of procedure.

If $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots\dots,$$

each of these ratios $=\left(\dfrac{pa^n+qc^n+re^n+\dots}{pb^n+qd^n+rf^n+\dots}\right)^{\frac{1}{n}}$,

where p, q, r, n *are any quantities whatever.*

Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots=k;$$

then $$a=bk,\ c=dk,\ e=fk,\dots;$$

whence $$pa^n=pb^nk^n,\ qc^n=qd^nk^n,\ re^n=rf^nk^n,\dots;$$

$$\therefore \frac{pa^n+qc^n+re^n+\dots}{pb^n+qd^n+rf^n+\dots}=\frac{pb^nk^n+qd^nk^n+rf^nk^n+\dots}{pb^n+qd^n+rf^n+\dots}$$

$$=k^n;$$

$$\therefore \left(\frac{pa^n+qc^n+re^n+\dots}{pb^n+qd^n+rf^n+\dots}\right)^{\frac{1}{n}}=k=\frac{a}{b}=\frac{c}{d}=\dots$$

By giving different values to p, q, r, n many particular cases of this general proposition may be deduced; or they may be proved independently by using the same method. For instance,

if $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots,$$

each of these ratios $=\dfrac{a+c+e+\dots}{b+d+f+\dots}$;

a result of such frequent utility that the following verbal equivalent should be noticed: *When a series of fractions are equal, each of them is equal to the sum of all the numerators divided by the sum of all the denominators.*

Example 1. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that

$$\frac{a^3b+2c^2e-3ae^2f}{b^4+2d^2f-3bf^3}=\frac{ace}{bdf}.$$

Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k;$$

then $$a=bk,\ c=dk,\ e=fk;$$

$$\therefore \frac{a^3b + 2c^2e - 3ae^2f}{b^4 + 2d^2f - 3bf^3} = \frac{b^4k^3 + 2d^2fk^3 - 3bf^3k^3}{b^4 + 2d^2f - 3bf^3}$$

$$= k^3 = \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f}$$

$$= \frac{ace}{bdf}.$$

Example 2. If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, prove that

$$\frac{x^2+a^2}{x+a} + \frac{y^2+b^2}{y+b} + \frac{z^2+c^2}{z+c} = \frac{(x+y+z)^2 + (a+b+c)^2}{x+y+z+a+b+c}.$$

Let $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k$, so that $x = ak$, $y = bk$, $z = ck$;

then $$\frac{x^2+a^2}{x+a} = \frac{a^2k^2+a^2}{ak+a} = \frac{(k^2+1)\,a}{k+1};$$

$$\therefore \frac{x^2+a^2}{x+a} + \frac{y^2+b^2}{y+b} + \frac{z^2+c^2}{z+c} = \frac{(k^2+1)\,a}{k+1} + \frac{(k^2+1)\,b}{k+1} + \frac{(k^2+1)\,c}{k+1}$$

$$= \frac{(k^2+1)\,(a+b+c)}{k+1}$$

$$= \frac{k^2\,(a+b+c)^2 + (a+b+c)^2}{k\,(a+b+c) + a+b+c}$$

$$= \frac{(ka+kb+kc)^2 + (a+b+c)^2}{(ka+kb+kc) + a+b+c}$$

$$= \frac{(x+y+z)^2 + (a+b+c)^2}{x+y+z+a+b+c}.$$

13. If an equation is homogeneous with respect to certain quantities, we may for these quantities substitute in the equation any others proportional to them. For instance, the equation

$$lx^3y + mxy^2z + ny^2z^2 = 0$$

is homogeneous in x, y, z. Let α, β, γ be three quantities proportional to x, y, z respectively.

Put $k = \frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$, so that $x = \alpha k$, $y = \beta k$, $z = \gamma k$;

then $$l\alpha^3\beta k^4 + m\alpha\beta^2\gamma k^4 + n\beta^2\gamma^2k^4 = 0,$$

that is, $$l\alpha^3\beta + m\alpha\beta^2\gamma + n\beta^2\gamma^2 = 0;$$

an equation of the same form as the original one, but with α, β, γ in the places of x, y, z respectively.

14. The following theorem is important.

If $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots \frac{a_n}{b_n}$ *be unequal fractions, of which the denominators are all of the same sign, then the fraction*

$$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{b_1 + b_2 + b_3 + \ldots + b_n}$$

lies in magnitude between the greatest and least of them.

Suppose that all the denominators are positive. Let $\frac{a_r}{b_r}$ be the least fraction, and denote it by k; then

$$\frac{a_r}{b_r} = k; \qquad \therefore\ a_r = kb_r;$$

$$\frac{a_1}{b_1} > k; \qquad \therefore\ a_1 > kb_1;$$

$$\frac{a_2}{b_2} > k; \qquad \therefore\ a_2 > kb_2;$$

and so on;

$\therefore$ by addition,

$$a_1 + a_2 + a_3 + \ldots\ldots + a_n > (b_1 + b_2 + b_3 + \ldots\ldots + b_n)\, k;$$

$$\therefore\ \frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{b_1 + b_2 + b_3 + \ldots\ldots + b_n} > k; \text{ that is, } > \frac{a_r}{b_r}.$$

Similarly we may prove that

$$\frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{b_1 + b_2 + b_3 + \ldots\ldots + b_n} < \frac{a_s}{b_s},$$

where $\frac{a_s}{b_s}$ is the greatest of the given fractions.

In like manner the theorem may be proved when all the denominators are negative.

15. The ready application of the *general principle* involved in Art. 12 is of such great value in all branches of mathematics, that the student should be able to use it with some freedom in any particular case that may arise, without necessarily introducing an auxiliary symbol.

Example 1. If $\frac{x}{b+c-a} = \frac{y}{c+a-b} = \frac{z}{a+b-c}$,

prove that $$\frac{x+y+z}{a+b+c} = \frac{x(y+z)+y(z+x)+z(x+y)}{2(ax+by+cz)}.$$

$$\text{Each of the given fractions} = \frac{\text{sum of numerators}}{\text{sum of denominators}}$$

$$= \frac{x+y+z}{a+b+c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, if we multiply both numerator and denominator of the three given fractions by $y+z$, $z+x$, $x+y$ respectively,

$$\text{each fraction} = \frac{x(y+z)}{(y+z)(b+c-a)} = \frac{y(z+x)}{(z+x)(c+a-b)} = \frac{z(x+y)}{(x+y)(a+b-c)}$$

$$= \frac{\text{sum of numerators}}{\text{sum of denominators}}$$

$$= \frac{x(y+z)+y(z+x)+z(x+y)}{2ax+2by+2cz} \quad \ldots\ldots (2).$$

$\therefore$ from (1) and (2),

$$\frac{x+y+z}{a+b+c} = \frac{x(y+z)+y(z+x)+z(x+y)}{2(ax+by+cz)}.$$

Example 2. If $\dfrac{x}{l(mb+nc-la)} = \dfrac{y}{m(nc+la-mb)} = \dfrac{z}{n(la+mb-nc)}$,

prove that $\dfrac{l}{x(by+cz-ax)} = \dfrac{m}{y(cz+ax-by)} = \dfrac{n}{z(ax+by-cz)}$.

We have
$$\frac{\dfrac{x}{l}}{mb+nc-la} = \frac{\dfrac{y}{m}}{nc+la-mb} = \frac{\dfrac{z}{n}}{la+mb-nc}$$

$$= \frac{\dfrac{y}{m}+\dfrac{z}{n}}{2la}$$

$$= \text{two similar expressions};$$

$$\therefore \quad \frac{ny+mz}{a} = \frac{lz+nx}{b} = \frac{mx+ly}{c}.$$

Multiply the first of these fractions above and below by x, the second by y, and the third by z; then

$$\frac{nxy+mxz}{ax} = \frac{lyz+nxy}{by} = \frac{mxz+lyz}{cz}$$

$$= \frac{2lyz}{by+cz-ax}$$

$$= \text{two similar expressions};$$

$$\therefore \quad \frac{l}{x(by+cz-ax)} = \frac{m}{y(cz+ax-by)} = \frac{n}{z(ax+by-cz)}.$$

16. If we have *two* equations containing *three* unknown quantities in the first degree, such as

$$a_1x + b_1y + c_1z = 0 \quad\text{.......................}(1),$$
$$a_2x + b_2y + c_2z = 0 \quad\text{.......................}(2),$$

we cannot solve these completely; but by writing them in the form

$$a_1\left(\frac{x}{z}\right) + b_1\left(\frac{y}{z}\right) + c_1 = 0,$$

$$a_2\left(\frac{x}{z}\right) + b_2\left(\frac{y}{z}\right) + c_2 = 0,$$

we can, by regarding $\frac{x}{z}$ and $\frac{y}{z}$ as the unknowns, solve in the ordinary way and obtain

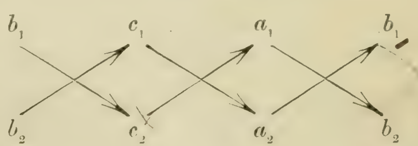
$$\frac{x}{z} = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad \frac{y}{z} = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1};$$

or, more symmetrically,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1} \quad\text{............}(3).$$

It thus appears that when we have two equations of the type represented by (1) and (2) we may always by the above formula write down the *ratios* $x : y : z$ in terms of the coefficients of the equations by the following rule:

Write down the coefficients of x, y, z in order, beginning with those of y; and repeat these as in the diagram.



Multiply the coefficients across in the way indicated by the arrows, remembering that in forming the products any one obtained by descending is positive, and any one obtained by ascending is negative. The three results

$$b_1c_2 - b_2c_1, \quad c_1a_2 - c_2a_1, \quad a_1b_2 - a_2b_1$$

are proportional to x, y, z respectively.

This is called the **Rule of Cross Multiplication.**

Example 1. Find the ratios of $x : y : z$ from the equations

$$7x = 4y + 8z, \quad 3z = 12x + 11y.$$

By transposition we have $7x - 4y - 8z = 0$,

$$12x + 11y - 3z = 0.$$

Write down the coefficients, thus

$$\begin{array}{cccc} -4 & -8 & 7 & -4 \\ 11 & -3 & 12 & 11, \end{array}$$

whence we obtain the products

$$(-4) \times (-3) - 11 \times (-8), \quad (-8) \times 12 - (-3) \times 7, \quad 7 \times 11 - 12 \times (-4),$$

or $$100, \quad -75, \quad 125;$$

$$\therefore \frac{x}{100} = \frac{y}{-75} = \frac{z}{125},$$

that is, $$\frac{x}{4} = \frac{y}{-3} = \frac{z}{5}.$$

Example 2. Eliminate x, y, z from the equations

$$a_1x + b_1y + c_1z = 0 \quad \text{.........(1)},$$

$$a_2x + b_2y + c_2z = 0 \quad \text{.........(2)},$$

$$a_3x + b_3y + c_3z = 0 \quad \text{.........(3)}.$$

From (2) and (3), by cross multiplication,

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{z}{a_2b_3 - a_3b_2};$$

denoting each of these ratios by k, by multiplying up, substituting in (1), and dividing out by k, we obtain

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0.$$

This relation is called the **eliminant** of the given equations.

Example 3. Solve the equations

$$ax + by + cz = 0 \quad \text{.........(1)},$$

$$x + y + z = 0 \quad \text{.........(2)},$$

$$bcx + cay + abz = (b - c)(c - a)(a - b) \quad \text{.........(3)}.$$

From (1) and (2), by cross multiplication,

$$\frac{x}{b - c} = \frac{y}{c - a} = \frac{z}{a - b} = k, \text{ suppose};$$

$$\therefore x = k(b - c), \; y = k(c - a), \; z = k(a - b).$$

Substituting in (3),

$$k\{bc(b - c) + ca(c - a) + ab(a - b)\} = (b - c)(c - a)(a - b),$$

$$k\{-(b - c)(c - a)(a - b)\} = (b - c)(c - a)(a - b);$$

$$\therefore k = -1;$$

whence $$x = c - b, \; y = a - c, \; z = b - a.$$

17. If in Art. 16 we put $z=1$, equations (1) and (2) become

$$a_1x+b_1y+c_1=0,$$
$$a_2x+b_2y+c_2=0;$$

and (3) becomes

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1};$$

or

$$x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\quad y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

Hence any two simultaneous equations involving two unknowns in the first degree may be solved by the rule of cross multiplication.

Example. Solve $5x-3y-1=0$, $x+2y=12$.

By transposition,

$$5x-3y-\ \ 1=0,$$
$$x+2y-12=0;$$

$$\therefore\ \frac{x}{36+2}=\frac{y}{-1+60}=\frac{1}{10+3};$$

whence

$$x=\frac{38}{13},\quad y=\frac{59}{13}.$$

EXAMPLES. I.

1. Find the ratio compounded of

(1) the ratio $2a : 3b$, and the duplicate ratio of $9b^2 : ab$.

(2) the subduplicate ratio of $64 : 9$, and the ratio $27 : 56$.

(3) the duplicate ratio of $\dfrac{2a}{b} : \dfrac{\sqrt{6a^2}}{b^2}$, and the ratio $3ax : 2by$.

2. If $x+7 : 2(x+14)$ in the duplicate ratio of $5 : 8$, find x.

3. Find two numbers in the ratio of $7 : 12$ so that the greater exceeds the less by 275.

4. What number must be added to each term of the ratio $5 : 37$ to make it equal to $1 : 3$?

5. If $x : y=3 : 4$, find the ratio of $7x-4y : 3x+y$.

6. If $15(2x^2-y^2)=7xy$, find the ratio of $x : y$.

7. If
$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f},$$
prove that
$$\frac{2a^4b^2+3a^2e^2-5e^4f}{2b^6+3b^2f^2-5f^5}=\frac{a^4}{b^4}.$$

8. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $\frac{a}{d}$ is equal to
$$\sqrt{\frac{a^5+b^2c^2+a^3c^2}{b^4c+d^4+b^2cd^2}}.$$

9. If
$$\frac{x}{q+r-p}=\frac{y}{r+p-q}=\frac{z}{p+q-r},$$
shew that
$$(q-r)x+(r-p)y+(p-q)z=0.$$

10. If $\frac{y}{x-z}=\frac{y+x}{z}=\frac{x}{y}$, find the ratios of $x : y : z$.

11. If
$$\frac{y+z}{pb+qc}=\frac{z+x}{pc+qa}=\frac{x+y}{pa+qb},$$
shew that
$$\frac{2(x+y+z)}{a+b+c}=\frac{(b+c)x+(c+a)y+(a+b)z}{bc+ca+ab}.$$

12. If
$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c},$$
shew that
$$\frac{x^3+a^3}{x^2+a^2}+\frac{y^3+b^3}{y^2+b^2}+\frac{z^3+c^3}{z^2+c^2}=\frac{(x+y+z)^3+(a+b+c)^3}{(x+y+z)^2+(a+b+c)^2}.$$

13. If
$$\frac{2y+2z-x}{a}=\frac{2z+2x-y}{b}=\frac{2x+2y-z}{c},$$
shew that
$$\frac{x}{2b+2c-a}=\frac{y}{2c+2a-b}=\frac{z}{2a+2b-c}.$$

14. If
$$(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2,$$
shew that
$$x : a=y : b=z : c.$$

15. If $l(my+nz-lx)=m(nz+lx-my)=n(lx+my-nz)$,

prove
$$\frac{y+z-x}{l}=\frac{z+x-y}{m}=\frac{x+y-z}{n}.$$

16. Shew that the eliminant of
$$ax+cy+bz=0,\ cx+by+az=0,\ bx+ay+cz=0,$$
is
$$a^3+b^3+c^3-3abc=0.$$

17. Eliminate x, y, z from the equations
$$ax+hy+gz=0,\ hx+by+fz=0,\ gx+fy+cz=0.$$

18. If
$$x=cy+bz, \ y=az+cx, \ z=bx+ay,$$
shew that
$$\frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}=\frac{z^2}{1-c^2}.$$

19. Given that $a(y+z)=x$, $b(z+x)=y$, $c(x+y)=z$,
prove that
$$bc+ca+ab+2abc=1.$$

Solve the following equations:

20.
$$3x-4y+7z=0,$$
$$2x-y-2z=0,$$
$$3x^3-y^3+z^3=18.$$

21.
$$x+y=z,$$
$$3x-2y+17z=0,$$
$$x^3+3y^3+2z^3=167.$$

22.
$$7yz+3zx=4xy,$$
$$21yz-3zx=4xy,$$
$$x+2y+3z=19.$$

23.
$$3x^2-2y^2+5z^2=0,$$
$$7x^2-3y^2-15z^2=0,$$
$$5x-4y+7z=6.$$

24. If
$$\frac{l}{\sqrt{a}-\sqrt{b}}+\frac{m}{\sqrt{b}-\sqrt{c}}+\frac{n}{\sqrt{c}-\sqrt{a}}=0,$$
$$\frac{l}{\sqrt{a}+\sqrt{b}}+\frac{m}{\sqrt{b}+\sqrt{c}}+\frac{n}{\sqrt{c}+\sqrt{a}}=0,$$
shew that
$$\frac{l}{(a-b)(c-\sqrt{ab})}=\frac{m}{(b-c)(a-\sqrt{bc})}=\frac{n}{(c-a)(b-\sqrt{ac})}.$$

Solve the equations:

25.
$$ax+by+cz=0,$$
$$bcx+cay+abz=0,$$
$$xyz+abc(a^3x+b^3y+c^3z)=0.$$

26.
$$ax+by+cz=a^2x+b^2y+c^2z=0,$$
$$x+y+z+(b-c)(c-a)(a-b)=0.$$

27. If
$$a(y+z)=x, \ b(z+x)=y, \ c(x+y)=z,$$
prove that
$$\frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ca)}=\frac{z^2}{c(1-ab)}.$$

28. If $ax+hy+gz=0$, $hx+by+fz=0$, $gx+fy+cz=0$;
prove that

(1) $$\frac{x^2}{bc-f^2}=\frac{y^2}{ca-g^2}=\frac{z^2}{ab-h^2}.$$

(2) $$(bc-f^2)(ca-g^2)(ab-h^2)=(fg-ch)(gh-af)(hf-bg).$$

CHAPTER II.

PROPORTION.

18. Definition. When two ratios are equal, the four quantities composing them are said to be **proportionals.** Thus if $\frac{a}{b} = \frac{c}{d}$, then a, b, c, d are proportionals. This is expressed by saying that a is to b as c is to d, and the proportion is written

$$a : b :: c : d;$$

or

$$a : b = c : d.$$

The terms a and d are called the *extremes*, b and c the *means*.

19. *If four quantities are in proportion, the product of the extremes is equal to the product of the means.*

Let a, b, c, d be the proportionals.

Then by definition $\frac{a}{b} = \frac{c}{d}$;

whence $ad = bc$.

Hence if any three terms of a proportion are given, the fourth may be found. Thus if a, c, d are given, then $b = \frac{ad}{c}$.

Conversely, if there are any four quantities, a, b, c, d, such that $ad = bc$, then a, b, c, d are proportionals; a and d being the extremes, b and c the means; or vice versâ.

20. Definition. Quantities are said to be in **continued proportion** when the first is to the second, as the second is to the third, as the third to the fourth; and so on. Thus a, b, c, d,...... are in continued proportion when

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \ldots\ldots\ldots$$

If three quantities a, b, c are in continued proportion, then

$$a : b = b : c;$$

$$\therefore \quad ac = b^2. \qquad \text{[Art. 18.]}$$

In this case b is said to be a **mean proportional** between a and c; and c is said to be a **third proportional** to a and b.

21. *If three quantities are proportionals the first is to the third in the duplicate ratio of the first to the second.*

Let the three quantities be a, b, c; then $\frac{a}{b} = \frac{b}{c}$.

Now $$\frac{a}{c} = \frac{a}{b} \times \frac{b}{c}$$

$$= \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2};$$

that is, $$a : c = a^2 : b^2.$$

It will be seen that this proposition is the same as the *definition* of duplicate ratio given in Euclid, Book V.

22. If $a : b = c : d$ and $e : f = g : h$, then will $ae : bf = cg : dh$.

For $$\frac{a}{b} = \frac{c}{d} \text{ and } \frac{e}{f} = \frac{g}{h};$$

$$\therefore \frac{ae}{bf} = \frac{cg}{dh},$$

or $$ae : bf = cg : dh.$$

Cor. If $$a : b = c : d,$$

and $$b : x = d : y,$$

then $$a : x = c : y.$$

This is the theorem known as *ex æquali* in Geometry.

23. If four quantities a, b, c, d form a proportion, many other proportions may be deduced by the properties of fractions. The results of these operations are very useful, and some of them are often quoted by the annexed names borrowed from Geometry.

(1) If $a : b = c : d$, then $b : a = d : c$. [*Invertendo.*]

For $\frac{a}{b} = \frac{c}{d}$; therefore $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$;

that is $$\frac{b}{a} = \frac{d}{c};$$

or $$b : a = d : c.$$

(2) If $a : b = c : d$, then $a : c = b : d$. [*Alternando.*]

For $ad = bc$; therefore $\frac{ad}{cd} = \frac{bc}{cd}$;

that is, $$\frac{a}{c} = \frac{b}{d};$$

or $$a : c = b : d.$$

(3) If $a : b = c : d$, then $a + b : b = c + d : d$. [*Componendo.*]

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} + 1 = \frac{c}{d} + 1$;

that is $$\frac{a+b}{b} = \frac{c+d}{d};$$

or $$a + b : b = c + d : d.$$

(4) If $a : b = c : d$, then $a - b : b = c - d : d$. [*Dividendo.*]

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} - 1 = \frac{c}{d} - 1$;

that is, $$\frac{a-b}{b} = \frac{c-d}{d};$$

or $$a - b : b = c - d : d.$$

(5) If $a : b = c : d$, then $a + b : a - b = c + d : c - d$

For by (3) $$\frac{a+b}{b} = \frac{c+d}{d};$$

and by (4) $$\frac{a-b}{b} = \frac{c-d}{d};$$

$\therefore$ by division, $$\frac{a+b}{a-b} = \frac{c+d}{c-d};$$

or $$a + b : a - b = c + d : c - d.$$

This proposition is usually quoted as *Componendo and Dividendo.*

Several other proportions may be proved in a similar way.

24. The results of the preceding article are the algebraical equivalents of some of the propositions in the fifth book of Euclid, and the student is advised to make himself familiar with them in their verbal form. For example, *dividendo* may be quoted as follows:

When there are four proportionals, the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth.

25. We shall now compare the algebraical definition of proportion with that given in Euclid.

Euclid's definition is as follows:

Four quantities are said to be proportionals when if *any equimultiples whatever* be taken of the first and third, and also *any equimultiples whatever* of the second and fourth, the multiple of the third is greater than, equal to, or less than the multiple of the fourth, according as the multiple of the first is greater than, equal to, or less than the multiple of the second.

In algebraical symbols the definition may be thus stated:

Four quantities a, b, c, d are in proportion when $pc \gtreqless qd$ according as $pa \gtreqless qb$, p and q being *any positive integers whatever.*

I. To deduce the geometrical definition of proportion from the algebraical definition.

Since $\frac{a}{b} = \frac{c}{d}$, by multiplying both sides by $\frac{p}{q}$, we obtain

$$\frac{pa}{qb} = \frac{pc}{qd};$$

hence, from the properties of fractions,

$$pc \gtreqless qd \text{ according as } pa \gtreqless qb,$$

which proves the proposition.

II. To deduce the algebraical definition of proportion from the geometrical definition.

Given that $pc \gtreqless qd$ according as $pa \gtreqless qb$, to prove

$$\frac{a}{b} = \frac{c}{d}.$$

If $\frac{a}{b}$ is not equal to $\frac{c}{d}$, one of them must be the greater. Suppose $\frac{a}{b} > \frac{c}{d}$; then it will be possible to find some fraction $\frac{q}{p}$ which lies between them, q and p being positive integers.

Hence $$\frac{a}{b} > \frac{q}{p} \quad\quad\quad (1),$$

and $$\frac{c}{d} < \frac{q}{p} \quad\quad\quad (2).$$

From (1) $\quad pa > qb$;

from (2) $\quad pc < qd$;

and these contradict the hypothesis.

Therefore $\frac{a}{b}$ and $\frac{c}{d}$ are not unequal; that is $\frac{a}{b} = \frac{c}{d}$; which proves the proposition.

26. It should be noticed that the geometrical definition of proportion deals with *concrete* magnitudes, such as lines or areas, represented geometrically but not referred to any common unit of measurement. So that Euclid's definition is applicable to incommensurable as well as to commensurable quantities; whereas the algebraical definition, strictly speaking, applies only to commensurable quantities, since it tacitly assumes that a is the same determinate multiple, part, or parts, of b that c is of d. But the proofs which have been given for commensurable quantities will still be true for incommensurables, since the ratio of two incommensurables can always be made to differ from the ratio of two integers *by less than any assignable quantity*. This has been shewn in Art. 7; it may also be proved more generally as in the next article.

27. Suppose that a and b are incommensurable; divide b into m equal parts each equal to β, so that $b = m\beta$, where m is a positive integer. Also suppose β is contained in a more than n times and less than $n + 1$ times;

then $$\frac{a}{b} > \frac{n\beta}{m\beta} \text{ and } < \frac{(n+1)\beta}{m\beta},$$

that is, $$\frac{a}{b} \text{ lies between } \frac{n}{m} \text{ and } \frac{n+1}{m};$$

so that $\frac{a}{b}$ differs from $\frac{n}{m}$ by a quantity less than $\frac{1}{m}$. And since we

can choose β (our unit of measurement) as small as we please, m can be made as great as we please. Hence $\frac{1}{m}$ can be made as small as we please, and two integers n and m can be found whose ratio will express that of a and b to any required degree of accuracy.

28. The propositions proved in Art. 23 are often useful in solving problems. In particular, the solution of certain equations is greatly facilitated by a skilful use of the operations *componendo* and *dividendo*.

Example 1.
If $(2ma + 6mb + 3nc + 9nd)(2ma - 6mb - 3nc + 9nd)$
$$= (2ma - 6mb + 3nc - 9nd)(2ma + 6mb - 3nc - 9nd),$$
prove that a, b, c, d are proportionals.

We have $$\frac{2ma + 6mb + 3nc + 9nd}{2ma - 6mb + 3nc - 9nd} = \frac{2ma + 6mb - 3nc - 9nd}{2ma - 6mb - 3nc + 9nd};$$

$\therefore$ componendo and dividendo,
$$\frac{2(2ma + 3nc)}{2(6mb + 9nd)} = \frac{2(2ma - 3nc)}{2(6mb - 9nd)}.$$

Alternando, $$\frac{2ma + 3nc}{2ma - 3nc} = \frac{6mb + 9nd}{6mb - 9nd}.$$

Again, componendo and dividendo,
$$\frac{4ma}{6nc} = \frac{12mb}{18nd};$$

whence $$\frac{a}{c} = \frac{b}{d},$$

or $$a : b = c : d.$$

Example 2. Solve the equation
$$\frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} = \frac{4x-1}{2}.$$

We have, componendo and dividendo,
$$\frac{\sqrt{x+1}}{\sqrt{x-1}} = \frac{4x+1}{4x-3};$$
$$\therefore \frac{x+1}{x-1} = \frac{16x^2 + 8x + 1}{16x^2 - 24x + 9}.$$

Again, componendo and dividendo,
$$\frac{2x}{2} = \frac{32x^2 - 16x + 10}{32x - 8},$$
$$\therefore x = \frac{16x^2 - 8x + 5}{16x - 4};$$

whence $$16x^2 - 4x = 16x^2 - 8x + 5;$$
$$\therefore x = \frac{5}{4}.$$

EXAMPLES. II.

1. Find the fourth proportional to 3, 5, 27.

2. Find the mean proportional between

(1) 6 and 24, (2) $360a^4$ and $250a^2b^2$.

3. Find the third proportional to $\frac{x}{y}+\frac{y}{x}$ and $\frac{x}{y}$.

If $a : b = c : d$, prove that

4. $a^2c+ac^2 : b^2d+bd^2 = (a+c)^3 : (b+d)^3$.

5. $pa^2+qb^2 : pa^2-qb^2 = pc^2+qd^2 : pc^2-qd^2$.

6. $a-c : b-d = \sqrt{a^2+c^2} : \sqrt{b^2+d^2}$.

7. $\sqrt{a^2+c^2} : \sqrt{b^2+d^2} = \sqrt{ac+\frac{c^3}{a}} : \sqrt{bd+\frac{d^3}{b}}$.

If a, b, c, d are in continued proportion, prove that

8. $a : b+d = c^3 : c^2d+d^3$.

9. $2a+3d : 3a-4d = 2a^3+3b^3 : 3a^3-4b^3$.

10. $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

11. If b is a mean proportional between a and c, prove that

$$\frac{a^2-b^2+c^2}{a^{-2}-b^{-2}+c^{-2}}=b^4.$$

12. If $a : b = c : d$, and $e : f = g : h$, prove that

$$ae+bf : ae-bf = cg+dh : cg-dh.$$

Solve the equations:

13. $\dfrac{2x^3-3x^2+x+1}{2x^3-3x^2-x-1}=\dfrac{3x^3-x^2+5x-13}{3x^3-x^2-5x+13}$.

14. $\dfrac{3x^4+x^2-2x-3}{3x^4-x^2+2x+3}=\dfrac{5x^4+2x^2-7x+3}{5x^4-2x^2+7x-3}$.

15. $\dfrac{(m+n)x-(a-b)}{(m-n)x-(a+b)}=\dfrac{(m+n)x+a+c}{(m-n)x+a-c}$.

16. If a, b, c, d are proportionals, prove that

$$a+d=b+c+\frac{(a-b)(a-c)}{a}.$$

17. If a, b, c, d, e are in continued proportion, prove that

$$(ab+bc+cd+de)^2=(a^2+b^2+c^2+d^2)(b^2+c^2+d^2+e^2).$$

18. If the work done by $x-1$ men in $x+1$ days is to the work done by $x+2$ men in $x-1$ days in the ratio of 9 : 10, find x.

19. Find four proportionals such that the sum of the extremes is 21, the sum of the means 19, and the sum of the squares of all four numbers is 442.

20. Two casks A and B were filled with two kinds of sherry, mixed in the cask A in the ratio of 2 : 7, and in the cask B in the ratio of 1 : 5. What quantity must be taken from each to form a mixture which shall consist of 2 gallons of one kind and 9 gallons of the other?

21. Nine gallons are drawn from a cask full of wine; it is then filled with water, then nine gallons of the mixture are drawn, and the cask is again filled with water. If the quantity of wine now in the cask be to the quantity of water in it as 16 to 9, how much does the cask hold?

22. If four positive quantities are in continued proportion, shew that the difference between the first and last is at least three times as great as the difference between the other two.

23. In England the population increased 15·9 per cent. between 1871 and 1881; if the town population increased 18 per cent. and the country population 4 per cent., compare the town and country populations in 1871.

24. In a certain country the consumption of tea is five times the consumption of coffee. If a per cent. more tea and b per cent. more coffee were consumed, the aggregate amount consumed would be $7c$ per cent. more; but if b per cent. more tea and a per cent. more coffee were consumed, the aggregate amount consumed would be $3c$ per cent. more: compare a and b.

25. Brass is an alloy of copper and zinc; bronze is an alloy containing 80 per cent. of copper, 4 of zinc, and 16 of tin. A fused mass of brass and bronze is found to contain 74 per cent. of copper, 16 of zinc, and 10 of tin: find the ratio of copper to zinc in the composition of brass.

26. A crew can row a certain course up stream in 84 minutes; they can row the same course down stream in 9 minutes less than they could row it in still water: how long would they take to row down with the stream?

CHAPTER III.

VARIATION.

29. Definition. One quantity A is said to **vary directly** as another B, when the two quantities depend upon each other in such a manner that if B is changed, A is changed *in the same ratio.*

Note. The word *directly* is often omitted, and A is said to vary as B.

For instance: if a train moving at a uniform rate travels 40 miles in 60 minutes, it will travel 20 miles in 30 minutes, 80 miles in 120 minutes, and so on; the distance in each case being increased or diminished in the same ratio as the time. This is expressed by saying that when the velocity is uniform *the distance is proportional to the time*, or *the distance varies as the time.*

30. The symbol $\propto$ is used to denote variation; so that $A \propto B$ is read "A varies as B."

31. *If* A *varies as* B, *then* A *is equal to* B *multiplied by some constant quantity.*

For suppose that $a, a_1, a_2, a_3 \ldots, b, b_1, b_2, b_3 \ldots$ are corresponding values of A and B.

Then, by definition, $\dfrac{a}{a_1} = \dfrac{b}{b_1}$; $\dfrac{a}{a_2} = \dfrac{b}{b_2}$; $\dfrac{a}{a_3} = \dfrac{b}{b_3}$; and so on,

$$\therefore \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \ldots, \text{ each being equal to } \frac{a}{b}.$$

Hence $\dfrac{\text{any value of } A}{\text{the corresponding value of } B}$ is always the same;

that is, $\dfrac{A}{B} = m$, where m is constant.

$$\therefore A = mB.$$

If any pair of corresponding values of A and B are known, the constant m can be determined. For instance, if $A = 3$ when $B = 12$,

we have $$3 = m \times 12;$$

$$\therefore m = \tfrac{1}{4},$$

and $$A = \tfrac{1}{4} B.$$

32. Definition. One quantity A is said to **vary inversely** as another B, when A varies *directly* as the reciprocal of B.

Thus if A varies inversely as B, $A = \dfrac{m}{B}$, where m is constant.

The following is an illustration of inverse variation: If 6 men do a certain work in 8 hours, 12 men would do the same work in 4 hours, 2 men in 24 hours; and so on. Thus it appears that when the number of men is increased, the time is proportionately decreased; and vice-versâ.

Example 1. The cube root of x varies inversely as the square of y; if $x = 8$ when $y = 3$, find x when $y = 1\tfrac{1}{2}$.

By supposition $\sqrt[3]{x} = \dfrac{m}{y^2}$, where m is constant.

Putting $x = 8$, $y = 3$, we have $2 = \dfrac{m}{9}$,

$$\therefore m = 18,$$

and $$\sqrt[3]{x} = \frac{18}{y^2};$$

hence, by putting $y = \dfrac{3}{2}$, we obtain $x = 512$.

Example 2. The square of the time of a planet's revolution varies as the cube of its distance from the Sun; find the time of Venus' revolution, assuming the distances of the Earth and Venus from the Sun to be $91\tfrac{1}{4}$ and 66 millions of miles respectively.

Let P be the periodic time measured in days, D the distance in millions of miles; we have $$P^2 \propto D^3,$$

or $$P^2 = kD^3,$$

where k is some constant.

For the Earth, $$365 \times 365 = k \times 91\tfrac{1}{4} \times 91\tfrac{1}{4} \times 91\tfrac{1}{4},$$

whence $$k = \frac{4 \times 4 \times 4}{365};$$

$$\therefore P^2 = \frac{4 \times 4 \times 4}{365} D^3.$$

For Venus, $P^2 = \frac{4 \times 4 \times 4}{365} \times 66 \times 66 \times 66$;

whence $P = 4 \times 66 \times \sqrt{\frac{264}{365}}$

$= 264 \times \sqrt{\cdot 7233}$, approximately,

$= 264 \times \cdot 85$

$= 224 \cdot 4$.

Hence the time of revolution is nearly $224\frac{1}{2}$ days.

33. Definition. One quantity is said to **vary jointly** as a number of others, when it varies directly as their product.

Thus A varies jointly as B and C, when $A = mBC$. For instance, the interest on a sum of money varies jointly as the principal, the time, and the rate per cent.

34. Definition. A is said to vary directly as B and inversely as C, when A varies as $\frac{B}{C}$.

35. *If* A *varies as* B *when* C *is constant, and* A *varies as* C *when* B *is constant, then will* A *vary as* BC *when both* B *and* C *vary.*

The variation of A depends partly on that of B and partly on that of C. Suppose these latter variations to take place separately, each in its turn producing its own effect on A; also let a, b, c be certain simultaneous values of A, B, C.

1. *Let C be constant* while B changes to b; then A must undergo a partial change and will assume some intermediate value a', where

$$\frac{A}{a'} = \frac{B}{b} \qquad \text{(1)}.$$

2. *Let B be constant*, that is, let it retain its value b, while C changes to c; then A must complete its change and pass from its intermediate value a' to its final value a, where

$$\frac{a'}{a} = \frac{C}{c} \qquad \text{(2)}.$$

From (1) and (2) $\frac{A}{a'} \times \frac{a'}{a} = \frac{B}{b} \times \frac{C}{c}$;

that is, $A = \frac{a}{bc} \cdot BC$,

or A varies as BC.

36. The following are illustrations of the theorem proved in the last article.

The amount of work done by *a given number of men* varies directly as the number of days they work, and the amount of work done *in a given time* varies directly as the number of men; therefore when the number of days and the number of men are both variable, the amount of work will vary as the product of the number of men and the number of days.

Again, in Geometry the area of a triangle varies directly as its base when the height is constant, and directly as the height when the base is constant; and when both the height and base are variable, the area varies as the product of the numbers representing the height and the base.

Example. The volume of a right circular cone varies as the square of the radius of the base when the height is constant, and as the height when the base is constant. If the radius of the base is 7 feet and the height 15 feet, the volume is 770 cubic feet; find the height of a cone whose volume is 132 cubic feet and which stands on a base whose radius is 3 feet.

Let h and r denote respectively the height and radius of the base measured in feet; also let V be the volume in cubic feet.

Then $V=mr^2h$, where m is constant.

By supposition, $$770=m\times 7^2\times 15;$$

whence $$m=\frac{22}{21};$$

$$\therefore\ V=\frac{22}{21}r^2h.$$

$\therefore$ by substituting $V=132$, $r=3$, we get

$$132=\frac{22}{21}\times 9\times h;$$

whence $$h=14;$$

and therefore the height is 14 feet.

37. The proposition of Art. 35 can easily be extended to the case in which the variation of A depends upon that of more than two variables. Further, the variations may be either direct or inverse. The principle is interesting because of its frequent occurrence in Physical Science. For example, in the theory of gases it is found by experiment that the pressure (p) of a gas varies as the "absolute temperature" (t) when its volume (v) is constant, and that the pressure varies inversely as the volume when the temperature is constant; that is

$$p \propto t, \text{ when } v \text{ is constant};$$

and $$p \propto \frac{1}{v}, \text{ when } t \text{ is constant.}$$

From these results we should expect that, when both t and v are variable, we should have the formula

$$p \propto \frac{t}{v}, \text{ or } pv = kt, \text{ where } k \text{ is constant;}$$

and by actual experiment this is found to be the case.

Example. The duration of a railway journey varies directly as the distance and inversely as the velocity; the velocity varies directly as the square root of the quantity of coal used per mile, and inversely as the number of carriages in the train. In a journey of 25 miles in half an hour with 18 carriages 10 cwt. of coal is required; how much coal will be consumed in a journey of 21 miles in 28 minutes with 16 carriages?

Let t be the time expressed in hours,
d the distance in miles,
v the velocity in miles per hour,
q the number of cwt. of coal used per mile,
c the number of carriages.

We have $$t \propto \frac{d}{v},$$

and $$v \propto \frac{\sqrt{q}}{c},$$

whence $$t \propto \frac{cd}{\sqrt{q}},$$

or $$t = \frac{kcd}{\sqrt{q}}, \text{ where } k \text{ is constant.}$$

Substituting the values given, we have $\left(\text{since } q = \frac{10}{25}\right)$

$$\frac{1}{2} = \frac{k \times 18 \times 25 \times 5}{\sqrt{10}};$$

that is, $$k = \frac{\sqrt{10}}{125 \times 36}.$$

Hence $$t = \frac{\sqrt{10} \cdot cd}{125 \times 36 \sqrt{q}}.$$

Substituting now the values of t, c, d given in the second part of the question, we have

$$\frac{28}{60} = \frac{\sqrt{10} \times 16 \times 21}{125 \times 36 \sqrt{q}};$$

that is, $$\sqrt{q} = \frac{\sqrt{10} \times 16 \times 21}{75 \times 28} = \frac{4}{25} \sqrt{10};$$

whence $$q = \frac{32}{125}.$$

Hence the quantity of coal is $\frac{21 \times 32}{125} = 5\frac{47}{125}$ cwt.

EXAMPLES. III.

1. If x varies as y, and $x=8$ when $y=15$, find x when $y=10$.

2. If P varies inversely as Q, and $P=7$ when $Q=3$, find P when $Q=2\frac{1}{3}$.

3. If the square of x varies as the cube of y, and $x=3$ when $y=4$, find the value of y when $x=\frac{1}{\sqrt{3}}$.

4. A varies as B and C jointly; if $A=2$ when $B=\frac{3}{5}$ and $C=\frac{10}{27}$, find C when $A=54$ and $B=3$.

5. If A varies as C, and B varies as C, then $A \pm B$ and $\sqrt{AB}$ will each vary as C.

6. If A varies as BC, then B varies inversely as $\frac{C}{A}$.

7. P varies directly as Q and inversely as R; also $P=\frac{2}{3}$ when $Q=\frac{3}{7}$ and $R=\frac{9}{14}$: find Q when $P=\sqrt{48}$ and $R=\sqrt{75}$.

8. If x varies as y, prove that x^2+y^2 varies as x^2-y^2.

9. If y varies as the sum of two quantities, of which one varies directly as x and the other inversely as x; and if $y=6$ when $x=4$, and $y=3\frac{1}{3}$ when $x=3$; find the equation between x and y.

10. If y is equal to the sum of two quantities one of which varies as x directly, and the other as x^2 inversely; and if $y=19$ when $x=2$, or 3; find y in terms of x.

11. If A varies directly as the square root of B and inversely as the cube of C, and if $A=3$ when $B=256$ and $C=2$, find B when $A=24$ and $C=\frac{1}{2}$.

12. Given that $x+y$ varies as $z+\frac{1}{z}$, and that $x-y$ varies as $z-\frac{1}{z}$, find the relation between x and z, provided that $z=2$ when $x=3$ and $y=1$.

13. If A varies as B and C jointly, while B varies as D^2, and C varies inversely as A, shew that A varies as D.

14. If y varies as the sum of three quantities of which the first is constant, the second varies as x, and the third as x^2; and if $y=0$ when $x=1$, $y=1$ when $x=2$, and $y=4$ when $x=3$; find y when $x=7$.

15. When a body falls from rest its distance from the starting point varies as the square of the time it has been falling: if a body falls through $402\frac{1}{2}$ feet in 5 seconds, how far does it fall in 10 seconds? Also how far does it fall in the 10^{th} second?

16. Given that the volume of a sphere varies as the cube of its radius, and that when the radius is $3\frac{1}{2}$ feet the volume is $179\frac{2}{3}$ cubic feet, find the volume when the radius is 1 foot 9 inches.

17. The weight of a circular disc varies as the square of the radius when the thickness remains the same; it also varies as the thickness when the radius remains the same. Two discs have their thicknesses in the ratio of 9 : 8; find the ratio of their radii if the weight of the first is twice that of the second.

18. At a certain regatta the number of races on each day varied jointly as the number of days from the beginning and end of the regatta up to and including the day in question. On three successive days there were respectively 6, 5 and 3 races. Which days were these, and how long did the regatta last?

19. The price of a diamond varies as the square of its weight. Three rings of equal weight, each composed of a diamond set in gold, have values £a, £b, £c, the diamonds in them weighing 3, 4, 5 carats respectively. Shew that the value of a diamond of one carat is

$$£\left(\frac{a+c}{2}-b\right),$$

the cost of workmanship being the same for each ring.

20. Two persons are awarded pensions in proportion to the square root of the number of years they have served. One has served 9 years longer than the other and receives a pension greater by £50. If the length of service of the first had exceeded that of the second by $4\frac{1}{4}$ years their pensions would have been in the proportion of 9 : 8. How long had they served and what were their respective pensions?

21. The attraction of a planet on its satellites varies directly as the mass (M) of the planet, and inversely as the square of the distance (D); also the square of a satellite's time of revolution varies directly as the distance and inversely as the force of attraction. If m_1, d_1, t_1, and m_2, d_2, t_2, are simultaneous values of M, D, T respectively, prove that

$$\frac{m_1 t_1^2}{m_2 t_2^2}=\frac{d_1^3}{d_2^3}.$$

Hence find the time of revolution of that moon of Jupiter whose distance is to the distance of our Moon as 35 : 31, having given that the mass of Jupiter is 343 times that of the Earth, and that the Moon's period is 27·32 days.

22. The consumption of coal by a locomotive varies as the square of the velocity; when the speed is 16 miles an hour the consumption of coal per hour is 2 tons: if the price of coal be 10*s.* per ton, and the other expenses of the engine be 11*s.* 3*d.* an hour, find the least cost of a journey of 100 miles.

CHAPTER IV.

ARITHMETICAL PROGRESSION.

38. Definition. Quantities are said to be in **Arithmetical Progression** when they increase or decrease by a *common difference.*

Thus each of the following series forms an Arithmetical Progression:

$$3,\ 7,\ 11,\ 15, \dots\dots\dots$$
$$8,\ 2,\ -4,\ -10, \dots\dots\dots$$
$$a,\ a+d,\ a+2d,\ a+3d,\ \dots\dots\dots$$

The common difference is found by subtracting *any* term of the series from that which *follows* it. In the first of the above examples the common difference is 4; in the second it is -6; in the third it is d.

39. If we examine the series

$$a,\ a+d,\ a+2d,\ a+3d, \dots$$

we notice that *in any term the coefficient of* d *is always less by one than the number of the term in the series.*

Thus the 3^{rd} term is $a+2d$;

6^{th} term is $a+5d$;

20^{th} term is $a+19d$;

and, generally, the p^{th} term is $a+(p-1)\,d$.

If n be the number of terms, and if l denote the last, or n^{th} term, we have $l=a+(n-1)\,d$.

40. *To find the sum of a number of terms in Arithmetical Progression.*

Let a denote the first term, d the common difference, and n the number of terms. Also let l denote the last term, and s

the required sum; then

$$s = a + (a + d) + (a + 2d) + \dots + (l - 2d) + (l - d) + l;$$

and, by writing the series in the reverse order,

$$s = l + (l - d) + (l - 2d) + \dots + (a + 2d) + (a + d) + a.$$

Adding together these two series,

$$2s = (a + l) + (a + l) + (a + l) + \dots \text{ to } n \text{ terms}$$
$$= n (a + l),$$

$$\therefore \; s = \frac{n}{2}(a + l) \quad\dots\dots(1);$$

and
$$l = a + (n - 1) d \quad\dots\dots(2),$$

$$\therefore \; s = \frac{n}{2}\{2a + (n - 1) d\} \quad\dots\dots(3).$$

41. In the last article we have three useful formulæ (1), (2), (3); in each of these any one of the letters may denote the unknown quantity when the three others are known. For instance, in (1) if we substitute given values for s, n, l, we obtain an equation for finding a; and similarly in the other formulæ. But it is necessary to guard against a too mechanical use of these general formulæ, and it will often be found better to solve simple questions by a mental rather than by an actual reference to the requisite formula.

Example 1. Find the sum of the series $5\frac{1}{2}$, $6\frac{3}{4}$, 8,......to 17 terms.

Here the common difference is $1\frac{1}{4}$; hence from (3),

the sum
$$= \frac{17}{2}\left\{2 \times \frac{11}{2} + 16 \times 1\tfrac{1}{4}\right\}$$
$$= \frac{17}{2}(11 + 20)$$
$$= \frac{17 \times 31}{2}$$
$$= 263\tfrac{1}{2}.$$

Example 2. The first term of a series is 5, the last 45, and the sum 400: find the number of terms, and the common difference.

If n be the number of terms, then from (1)

$$400 = \frac{n}{2}(5 + 45);$$

whence
$$n = 16.$$

If d be the common difference

$$45 = \text{the } 16^{\text{th}} \text{ term} = 5 + 15d;$$

whence $$d = 2\tfrac{2}{3}.$$

42. If *any two* terms of an Arithmetical Progression be given, the series can be completely determined; for the data furnish *two* simultaneous equations, the solution of which will give the first term and the common difference.

Example. The 54^{th} and 4^{th} terms of an A. P. are -61 and 64; find the 23^{rd} term.

If a be the first term, and d the common difference,

$$-61 = \text{the } 54^{\text{th}} \text{ term} = a + 53d;$$

and $$64 = \text{the } 4^{\text{th}} \text{ term} = a + 3d;$$

whence we obtain $$d = -\frac{5}{2},\ a = 71\tfrac{1}{2};$$

and the 23^{rd} term $= a + 22d = 16\tfrac{1}{2}$.

43. Definition. When three quantities are in Arithmetical Progression the middle one is said to be the **arithmetic mean** of the other two.

Thus a is the arithmetic mean between $a - d$ and $a + d$.

44. *To find the arithmetic mean between two given quantities.*

Let a and b be the two quantities; A the arithmetic mean. Then since a, A, b are in A. P. we must have

$$b - A = A - a,$$

each being equal to the common difference;

whence $$A = \frac{a + b}{2}.$$

45. Between two given quantities it is always possible to insert any number of terms such that the whole series thus formed shall be in A. P.; and by an extension of the definition in Art. 43, the terms thus inserted are called the *arithmetic means*.

Example. Insert 20 arithmetic means between 4 and 67.

Including the extremes, the number of terms will be 22; so that we have to find a series of 22 terms in A.P., of which 4 is the first and 67 the last.

Let d be the common difference;

then $$67 = \text{the } 22^{\text{nd}} \text{ term} = 4 + 21d;$$

whence $d = 3$, and the series is $4, 7, 10, \ldots\ldots 61, 64, 67$;

and the required means are $7, 10, 13, \ldots\ldots 58, 71, 64$.

46. *To insert a given number of arithmetic means between two given quantities.*

Let a and b be the given quantities, n the number of means.

Including the extremes the number of terms will be $n+2$; so that we have to find a series of $n+2$ terms in A.P., of which a is the first, and b is the last.

Let d be the common difference;

then $$b = \text{the } (n+2)^{\text{th}} \text{ term}$$
$$= a + (n+1)\,d;$$

whence $$d = \frac{b-a}{n+1};$$

and the required means are

$$a + \frac{b-a}{n+1}, \quad a + \frac{2\,(b-a)}{n+1}, \ldots\ldots a + \frac{n\,(b-a)}{n+1}.$$

Example 1. The sum of three numbers in A.P. is 27, and the sum of their squares is 293; find them.

Let a be the *middle* number, d the common difference; then the three numbers are $a-d,\ a,\ a+d$.

Hence $$a-d+a+a+d=27;$$

whence $a=9$, and the three numbers are $9-d,\ 9,\ 9+d$.

$$\therefore\ (9-d)^2+81+(9+d)^2=293;$$

whence $$d=\pm 5;$$

and the numbers are 4, 9, 14.

Example 2. Find the sum of the first p terms of the series whose n^{th} term is $3n-1$.

By putting $n=1$, and $n=p$ respectively, we obtain

$$\text{first term}=2, \quad \text{last term}=3p-1;$$

$$\therefore\ \text{sum}=\frac{p}{2}(2+3p-1)=\frac{p}{2}(3p+1).$$

EXAMPLES. IV. a.

1. Sum $2,\ 3\frac{1}{4},\ 4\frac{1}{2},\ldots$ to 20 terms.

2. Sum $49,\ 44,\ 39,\ldots$ to 17 terms.

3. Sum $\frac{3}{4},\ \frac{2}{3},\ \frac{7}{12},\ldots$ to 19 terms.

4. Sum $3, \frac{7}{3}, 1\frac{2}{3}, \ldots$ to n terms.

5. Sum $3{\cdot}75, 3{\cdot}5, 3{\cdot}25, \ldots$ to 16 terms.

6. Sum $-7\frac{1}{2}, -7, -6\frac{1}{2}, \ldots$ to 24 terms.

7. Sum $1{\cdot}3, -3{\cdot}1, -7{\cdot}5, \ldots$ to 10 terms.

8. Sum $\frac{6}{\sqrt{3}}, 3\sqrt{3}, \frac{12}{\sqrt{3}}, \ldots$ to 50 terms.

9. Sum $\frac{3}{\sqrt{5}}, \frac{4}{\sqrt{5}}, \sqrt{5}, \ldots$ to 25 terms.

10. Sum $a-3b, 2a-5b, 3a-7b, \ldots$ to 40 terms.

11. Sum $2a-b, 4a-3b, 6a-5b, \ldots$ to n terms.

12. Sum $\frac{a+b}{2}, a, \frac{3a-b}{2}, \ldots$ to 21 terms.

13. Insert 19 arithmetic means between $\frac{1}{4}$ and $-9\frac{3}{4}$.

14. Insert 17 arithmetic means between $3\frac{1}{2}$ and $-41\frac{1}{2}$.

15. Insert 18 arithmetic means between $-35x$ and $3x$.

16. Insert x arithmetic means between x^2 and 1.

17. Find the sum of the first n odd numbers.

18. In an A. P. the first term is 2, the last term 29, the sum 155; find the difference.

19. The sum of 15 terms of an A. P. is 600, and the common difference is 5; find the first term.

20. The third term of an A. P. is 18, and the seventh term is 30; find the sum of 17 terms.

21. The sum of three numbers in A. P. is 27, and their product is 504; find them.

22. The sum of three numbers in A. P. is 12, and the sum of their cubes is 408; find them.

23. Find the sum of 15 terms of the series whose n^{th} term is $4n+1$.

24. Find the sum of 35 terms of the series whose p^{th} term is $\frac{p}{7}+2$.

25. Find the sum of p terms of the series whose n^{th} term is $\frac{n}{a}+b$.

26. Find the sum of n terms of the series

$$\frac{2a^2-1}{a}, \quad 4a-\frac{3}{a}, \quad \frac{6a^2-5}{a}, \ldots$$

47. In an Arithmetical Progression when s, a, d are given, to determine the values of n we have the quadratic equation

$$s = \frac{n}{2}\left\{2a + (n-1)d\right\};$$

when both roots are positive and integral there is no difficulty in interpreting the result corresponding to each. In some cases a suitable interpretation can be given for a negative value of n.

Example. How many terms of the series -9, -6, -3,... must be taken that the sum may be 66?

Here $$\frac{n}{2}\{-18 + (n-1)\,3\} = 66;$$

that is, $$n^2 - 7n - 44 = 0,$$

or $$(n-11)(n+4) = 0;$$

$$\therefore\ n = 11 \text{ or } -4.$$

If we take 11 terms of the series, we have

$$-9,\ -6,\ -3,\ 0,\ 3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21;$$

the sum of which is 66.

If we begin at the *last* of these terms and *count backwards* four terms, the sum is also 66; and thus, although the negative solution does not directly answer the question proposed, we are enabled to give it an intelligible meaning, and we see that it answers a question closely connected with that to which the positive solution applies.

48. We can justify this interpretation in the general case in the following way.

The equation to determine n is

$$dn^2 + (2a - d)\,n - 2s = 0 \text{ .... } (1).$$

Since in the case under discussion the roots of this equation have opposite signs, let us denote them by n_1 and $-n_2$. The last term of the series corresponding to n_1 is

$$a + (n_1 - 1)\,d;$$

if we begin at this term and count *backwards*, the common difference must be denoted by $-d$, and the sum of n_2 terms is

$$\frac{n_2}{2}\left\{2\left(a + \overline{n_1 - 1}d\right) + (n_2 - 1)(-d)\right\},$$

and we shall shew that this is equal to s.

For the expression $= \dfrac{n_2}{2}\left\{2a + (2n_1 - n_2 - 1)\,d\right\}$

$$= \frac{1}{2}\left\{2an_2 + 2n_1n_2d - n_2(n_2+1)\,d\right\}$$

$$= \frac{1}{2}\left\{2n_1n_2d - (dn_2^2 - \overline{2a-d}\,.\,n_2)\right\}$$

$$= \frac{1}{2}(4s - 2s) = s,$$

since $-n_2$ satisfies $dn^2 + (2a-d)\,n - 2s = 0$, and $-n_1n_2$ is the product of the roots of this equation.

49. When the value of n is fractional there is no exact number of terms which corresponds to such a solution.

Example. How many terms of the series 26, 21, 16,...must be taken to amount to 74?

Here $\qquad \dfrac{n}{2}\{52 + (n-1)(-5)\} = 74;$

that is, $\qquad 5n^2 - 57n + 148 = 0,$

or $\qquad (n-4)(5n-37) = 0;$

$$\therefore\ n = 4 \text{ or } 7\tfrac{2}{5}.$$

Thus the number of terms is 4. It will be found that the sum of 7 terms is greater, while the sum of 8 terms is less than 74.

50. We add some Miscellaneous Examples.

Example 1. The sums of n terms of two arithmetic series are in the ratio of $7n+1 : 4n+27$; find the ratio of their 11th terms.

Let the first term and common difference of the two series be a_1, d_1 and a_2, d_2 respectively.

We have $\qquad \dfrac{2a_1 + (n-1)\,d_1}{2a_2 + (n-1)\,d_2} = \dfrac{7n+1}{4n+27}.$

Now we have to find the value of $\dfrac{a_1 + 10d_1}{a_2 + 10d_2}$; hence, by putting $n = 21$, we obtain

$$\frac{2a_1 + 20d_1}{2a_2 + 20d_2} = \frac{148}{111} = \frac{4}{3};$$

thus the required ratio is 4 : 3.

Example 2. If S_1, S_2, S_3,...S_p are the sums of n terms of arithmetic series whose first terms are 1, 2, 3, 4,... and whose common differences are 1, 3, 5, 7,...; find the value of

$$S_1 + S_2 + S_3 + \ldots + S_p.$$

We have
$$S_1=\frac{n}{2}\{2+(n-1)\}=\frac{n(n+1)}{2},$$
$$S_2=\frac{n}{2}\{4+(n-1)3\}=\frac{n(3n+1)}{2},$$
$$S_3=\frac{n}{2}\{6+(n-1)5\}=\frac{n(5n+1)}{2},$$
$$S_p=\frac{n}{2}\{2p+(n-1)(2p-1)\}=\frac{n}{2}\{(2p-1)n+1\};$$

$\therefore$ the required sum $=\frac{n}{2}\{(n+1)+(3n+1)+\ldots\ldots(\overline{2p-1}\,.\,n+1)\}$

$$=\frac{n}{2}\{(n+3n+5n+\ldots\overline{2p-1}\,.\,n)+p\}$$
$$=\frac{n}{2}\{n(1+3+5+\ldots\overline{2p-1})+p\}$$
$$=\frac{n}{2}(np^2+p)$$
$$=\frac{np}{2}(np+1).$$

EXAMPLES. IV. b.

1. Given $a=-2$, $d=4$ and $s=160$, find n.

2. How many terms of the series 12, 16, 20,... must be taken to make 208?

3. In an A. P. the third term is four times the first term, and the sixth term is 17; find the series.

4. The 2nd, 31st, and last terms of an A. P. are $7\frac{3}{4}$, $\frac{1}{2}$ and $-6\frac{1}{2}$ respectively; find the first term and the number of terms.

5. The 4th, 42nd, and last terms of an A. P. are 0, -95 and -125 respectively; find the first term and the number of terms.

6. A man arranges to pay off a debt of £3600 by 40 annual instalments which form an arithmetic series. When 30 of the instalments are paid he dies leaving a third of the debt unpaid: find the value of the first instalment.

7. Between two numbers whose sum is $2\frac{1}{6}$ an even number of arithmetic means is inserted; the sum of these means exceeds their number by unity: how many means are there?

8. The sum of n terms of the series 2, 5, 8,... is 950: find n.

9. Sum the series $\frac{1}{1+\sqrt{x}}$, $\frac{1}{1-x}$, $\frac{1}{1-\sqrt{x}}$, ... to n terms.

10. If the sum of 7 terms is 49, and the sum of 17 terms is 289, find the sum of n terms.

11. If the p^{th}, q^{th}, r^{th} terms of an A. P. are a, b, c respectively, shew that
$$(q-r)\,a+(r-p)\,b+(p-q)\,c=0.$$

12. The sum of p terms of an A. P. is q, and the sum of q terms is p; find the sum of $p+q$ terms.

13. The sum of four integers in A. P. is 24, and their product is 945; find them.

14. Divide 20 into four parts which are in A. P., and such that the product of the first and fourth is to the product of the second and third in the ratio of 2 to 3.

15. The p^{th} term of an A. P. is q, and the q^{th} term is p; find the m^{th} term.

16. How many terms of the series 9, 12, 15,... must be taken to make 306?

17. If the sum of n terms of an A. P. is $2n+3n^2$, find the r^{th} term.

18. If the sum of m terms of an A. P. is to the sum of n terms as m^2 to n^2, shew that the m^{th} term is to the n^{th} term as $2m-1$ is to $2n-1$.

19. Prove that the sum of an odd number of terms in A. P. is equal to the middle term multiplied by the number of terms.

20. If $s=n(5n-3)$ for all values of n, find the p^{th} term.

21. The number of terms in an A. P. is even; the sum of the odd terms is 24, of the even terms 30, and the last term exceeds the first by $10\frac{1}{2}$: find the number of terms.

22. There are two sets of numbers each consisting of 3 terms in A. P. and the sum of each set is 15. The common difference of the first set is greater by 1 than the common difference of the second set, and the product of the first set is to the product of the second set as 7 to 8: find the numbers.

23. Find the relation between x and y in order that the r^{th} mean between x and $2y$ may be the same as the r^{th} mean between $2x$ and y, n means being inserted in each case.

24. If the sum of an A. P. is the same for p as for q terms, shew that its sum for $p+q$ terms is zero.

CHAPTER V.

GEOMETRICAL PROGRESSION.

51. Definition. Quantities are said to be in **Geometrical Progression** when they increase or decrease by a *constant factor.*

Thus each of the following series forms a Geometrical Progression:

$$3,\ 6,\ 12,\ 24,\ \dots\dots\dots\dots\dots\dots$$

$$1,\quad -\frac{1}{3},\ \frac{1}{9},\quad -\frac{1}{27},\dots\dots\dots$$

$$a,\ ar,\ ar^2,\ ar^3,\ \dots\dots\dots\dots\dots\dots$$

The constant factor is also called the *common ratio*, and it is found by dividing *any* term by that which immediately *precedes* it. In the first of the above examples the common ratio is 2; in the second it is $-\frac{1}{3}$; in the third it is r.

52. If we examine the series

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \dots\dots$$

we notice that *in any term the index of* r *is always less by one than the number of the term in the series.*

Thus the 3rd term is ar^2;

the 6th term is ar^5;

the 20th term is ar^{19};

and, generally, the p^{th} term is ar^{p-1}.

If n be the number of terms, and if l denote the last, or n^{th} term, we have $l = ar^{n-1}$.

53. Definition. When three quantities are in Geometrical Progression the middle one is called the **geometric mean** between the other two.

To find the geometric mean between two given quantities.

Let a and b be the two quantities; G the geometric mean. Then since a, G, b are in G. P.,

$$\frac{b}{G} = \frac{G}{a},$$

each being equal to the common ratio;

$$\therefore G^2 = ab;$$

whence $$G = \sqrt{ab}.$$

54. *To insert a given number of geometric means between two given quantities.*

Let a and b be the given quantities, n the number of means.

In all there will be $n+2$ terms; so that we have to find a series of $n+2$ terms in G. P., of which a is the first and b the last.

Let r be the common ratio;

then $$b = \text{the } (n+2)^{\text{th}} \text{ term}$$
$$= ar^{n+1};$$
$$\therefore r^{n+1} = \frac{b}{a};$$
$$\therefore r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \quad \text{.................................} \quad (1).$$

Hence the required means are ar, ar^2,... ar^n, where r has the value found in (1).

Example. Insert 4 geometric means between 160 and 5.

We have to find 6 terms in G. P. of which 160 is the first, and 5 the sixth.

Let r be the common ratio;

$$\text{then } 5 = \text{the sixth term}$$
$$= 160r^5;$$
$$\therefore r^5 = \frac{1}{32};$$

whence $$r = \frac{1}{2};$$

and the means are 80, 40, 20, 10.

55. *To find the sum of a number of terms in Geometrical Progression.*

Let a be the first term, r the common ratio, n the number of terms, and s the sum required. Then

$$s = a + ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1};$$

multiplying every term by r, we have

$$rs = ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n.$$

Hence by subtraction,

$$rs - s = ar^n - a;$$

$$\therefore (r-1)s = a(r^n - 1);$$

$$\therefore s = \frac{a(r^n-1)}{r-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Changing the signs in numerator and denominator,

$$s = \frac{a(1-r^n)}{1-r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Note. It will be found convenient to remember both forms given above for s, using (2) in all cases except when r is positive and greater than 1.

Since $ar^{n-1} = l$, the formula (1) may be written

$$s = \frac{rl - a}{r - 1};$$

a form which is sometimes useful.

Example. Sum the series $\frac{2}{3}$, -1, $\frac{3}{2}$, ... to 7 terms.

The common ratio $= -\frac{3}{2}$; hence by formula (2)

the sum
$$= \frac{\frac{2}{3}\left\{1 - \left(-\frac{3}{2}\right)^7\right\}}{1 + \frac{3}{2}}$$

$$= \frac{\frac{2}{3}\left\{1 + \frac{2187}{128}\right\}}{\frac{5}{2}}$$

$$= \frac{2}{3} \times \frac{2315}{128} \times \frac{2}{5}$$

$$= \frac{463}{96}.$$

56. Consider the series $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots$

The sum to n terms $= \dfrac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$

$$= 2\left(1 - \frac{1}{2^n}\right)$$

$$= 2 - \frac{1}{2^{n-1}}.$$

From this result it appears that however many terms be taken the sum of the above series is always less than 2. Also we see that, by making n sufficiently large, we can make the fraction $\frac{1}{2^{n-1}}$ as small as we please. Thus by taking a sufficient number of terms the sum can be made to differ by as little as we please from 2.

In the next article a more general case is discussed.

57. From Art. 55 we have $s = \dfrac{a(1 - r^n)}{1 - r}$

$$= \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Suppose r is a proper fraction; then the greater the value of n the smaller is the value of r^n, and consequently of $\dfrac{ar^n}{1-r}$; and therefore by making n sufficiently large, we can make the sum of n terms of the series differ from $\dfrac{a}{1-r}$ by as small a quantity as we please.

This result is usually stated thus: *the sum of an infinite number of terms of a decreasing Geometrical Progression is* $\dfrac{a}{1-r}$; or more briefly, *the sum to infinity is* $\dfrac{a}{1-r}$.

Example 1. Find three numbers in G. P. whose sum is 19, and whose product is 216.

Denote the numbers by $\frac{a}{r}$, a, ar; then $\frac{a}{r} \times a \times ar = 216$; hence $a = 6$, and the numbers are $\frac{6}{r}$, 6, $6r$.

$$\therefore \frac{6}{r} + 6 + 6r = 19;$$

$$\therefore 6 - 13r + 6r^2 = 0;$$

whence $$r = \frac{3}{2} \text{ or } \frac{2}{3}.$$

Thus the numbers are 4, 6, 9.

Example 2. The sum of an infinite number of terms in G. P. is 15, and the sum of their squares is 45; find the series.

Let a denote the first term, r the common ratio; then the sum of the terms is $\frac{a}{1-r}$; and the sum of their squares is $\frac{a^2}{1-r^2}$.

Hence $$\frac{a}{1-r} = 15 \quad \ldots\ldots\ldots\ldots (1),$$

$$\frac{a^2}{1-r^2} = 45 \quad \ldots\ldots\ldots\ldots (2).$$

Dividing (2) by (1) $$\frac{a}{1+r} = 3 \quad \ldots\ldots\ldots\ldots (3),$$

and from (1) and (3) $$\frac{1+r}{1-r} = 5;$$

whence $r = \frac{2}{3}$, and therefore $a = 5$.

Thus the series is $5, \frac{10}{3}, \frac{20}{9}, \ldots\ldots$

EXAMPLES. V. a.

1. Sum $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \ldots$ to 7 terms.
2. Sum $-2, 2\frac{1}{2}, -3\frac{1}{8}, \ldots$ to 6 terms.
3. Sum $\frac{3}{4}, 1\frac{1}{2}, 3, \ldots$ to 8 terms.
4. Sum $2, -4, 8, \ldots$ to 10 terms.
5. Sum $16{\cdot}2, 5{\cdot}4, 1{\cdot}8, \ldots$ to 7 terms.
6. Sum $1, 5, 25, \ldots$ to p terms.
7. Sum $3, -4, \frac{16}{3}, \ldots$ to $2n$ terms.
8. Sum $1, \sqrt{3}, 3, \ldots$ to 12 terms.
9. Sum $\frac{1}{\sqrt{2}}, -2, \frac{8}{\sqrt{2}}, \ldots$ to 7 terms.

10. Sum $-\frac{1}{3}, \frac{1}{2}, -\frac{3}{4}, \ldots$ to 7 terms.

11. Insert 3 geometric means between $2\frac{1}{4}$ and $\frac{4}{9}$.

12. Insert 5 geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.

13. Insert 6 geometric means between 14 and $-\frac{7}{64}$.

Sum the following series to infinity:

14. $\frac{8}{5}, -1, \frac{5}{8}, \ldots$

15. $\cdot 45, \cdot 015, \cdot 0005, \ldots$

16. $1\cdot 665, -1\cdot 11, \cdot 74, \ldots$

17. $3^{-1}, 3^{-2}, 3^{-3}, \ldots$

18. $3, \sqrt{3}, 1, \ldots$

19. $7, \sqrt{42}, 6, \ldots$

20. The sum of the first 6 terms of a G. P. is 9 times the sum of the first 3 terms; find the common ratio.

21. The fifth term of a G. P. is 81, and the second term is 24; find the series.

22. The sum of a G. P. whose common ratio is 3 is 728, and the last term is 486; find the first term.

23. In a G. P. the first term is 7, the last term 448, and the sum 889; find the common ratio.

24. The sum of three numbers in G. P. is 38, and their product is 1728; find them.

25. The continued product of three numbers in G. P. is 216, and the sum of the products of them in pairs is 156; find the numbers.

26. If S_p denote the sum of the series $1 + r^p + r^{2p} + \ldots$ *ad inf.*, and s_p the sum of the series $1 - r^p + r^{2p} - \ldots$ *ad inf.*, prove that

$$S_p + s_p = 2S_{2p}.$$

27. If the p^{th}, q^{th}, r^{th} terms of a G. P. be a, b, c respectively, prove that

$$a^{q-r} b^{r-p} c^{p-q} = 1.$$

28. The sum of an infinite number of terms of a G. P. is 4, and the sum of their cubes is 192; find the series.

58. Recurring decimals furnish a good illustration of infinite Geometrical Progressions.

Example. Find the value of $\cdot 4\dot{2}\dot{3}$.

$$\cdot 4\dot{2}\dot{3} = \cdot 4232323\ldots\ldots$$

$$= \frac{4}{10} + \frac{23}{1000} + \frac{23}{100000} + \ldots\ldots$$

$$= \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \ldots\ldots;$$

that is, $$\cdot 4\dot{2}\dot{3} = \frac{4}{10} + \frac{23}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \dots\dots\right)$$

$$= \frac{4}{10} + \frac{23}{10^3} \cdot \frac{1}{1 - \frac{1}{10^2}}$$

$$= \frac{4}{10} + \frac{23}{10^3} \cdot \frac{100}{99}$$

$$= \frac{4}{10} + \frac{23}{990}$$

$$= \frac{419}{990},$$

which agrees with the value found by the usual arithmetical rule.

59. The general rule for reducing any recurring decimal to a vulgar fraction may be proved by the method employed in the last example; but it is easier to proceed as follows.

To find the value of a recurring decimal.

Let P denote the figures which do not recur, and suppose them p in number; let Q denote the recurring period consisting of q figures; let D denote the value of the recurring decimal; then

$$D = \cdot PQQQ\dots\dots\dots\dots ;$$

$$\therefore\ 10^p \times D = P\cdot QQQ\dots\dots\dots\dots ;$$

and $$10^{p+q} \times D = PQ\cdot QQQ\dots\dots\dots ;$$

therefore, by subtraction, $(10^{p+q} - 10^p)\,D = PQ - P$;

that is, $$10^p\,(10^q - 1)\,D = PQ - P;$$

$$\therefore\ D = \frac{PQ - P}{(10^q - 1)\,10^p}.$$

Now $10^q - 1$ is a number consisting of q nines; therefore the denominator consists of q nines followed by p ciphers. Hence we have the following rule for reducing a recurring decimal to a vulgar fraction:

For the numerator subtract the integral number consisting of the non-recurring figures from the integral number consisting of the non-recurring and recurring figures; for the denominator take a number consisting of as many nines as there are recurring figures followed by as many ciphers as there are non-recurring figures.

60. *To find the sum of* n *terms of the series*

$$a, \ (a+d)r, \ (a+2d)r^2, \ (a+3d)r^3, \ldots\ldots\ldots$$

in which each term is the product of corresponding terms in an arithmetic and geometric series.

Denote the sum by S; then

$$S = a + (a+d)r + (a+2d)r^2 + \ldots + (a+\overline{n-1}d)r^{n-1};$$

$$\therefore \ rS = \qquad ar + (a+d)r^2 + \ldots + (a+\overline{n-2}d)r^{n-1} + (a+\overline{n-1}d)r^n.$$

By subtraction,

$$S(1-r) = a + (dr + dr^2 + \ldots + dr^{n-1}) - (a+\overline{n-1}d)r^n$$

$$= a + \frac{dr(1-r^{n-1})}{1-r} - (a+\overline{n-1}d)r^n;$$

$$\therefore \ S = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+\overline{n-1}d)r^n}{1-r}.$$

Cor. Write S in the form

$$\frac{a}{1-r} + \frac{dr}{(1-r)^2} - \frac{dr^n}{(1-r)^2} - \frac{(a+\overline{n-1}d)r^n}{1-r};$$

then if $r<1$, we can make r^n as small as we please by taking n sufficiently great. In this case, *assuming that all the terms which involve* r^n *can be made so small that they may be neglected,* we obtain $\frac{a}{1-r} + \frac{dr}{(1-r)^2}$ for the sum to infinity. We shall refer to this point again in Chap. XXI.

In summing to infinity series of this class it is usually best to proceed as in the following example.

Example 1. If $x<1$, sum the series

$$1 + 2x + 3x^2 + 4x^3 + \ldots\ldots \text{ to infinity.}$$

Let
$$S = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots;$$

$$\therefore \ xS = \qquad x + 2x^2 + 3x^3 + \ldots\ldots;$$

$$\therefore \ S(1-x) = 1 + x + x^2 + x^3 + \ldots\ldots$$

$$= \frac{1}{1-x};$$

$$\therefore \ S = \frac{1}{(1-x)^2}.$$

Example 2. Sum the series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots$ to n terms.

Let
$$S=1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots\ldots+\frac{3n-2}{5^{n-1}};$$

$$\therefore\ \frac{1}{5}S=\quad\frac{1}{5}+\frac{4}{5^2}+\frac{7}{5^3}+\ldots\ldots+\frac{3n-5}{5^{n-1}}+\frac{3n-2}{5^n};$$

$$\therefore\ \frac{4}{5}S=1+\left(\frac{3}{5}+\frac{3}{5^2}+\frac{3}{5^3}+\ldots\ldots+\frac{3}{5^{n-1}}\right)-\frac{3n-2}{5^n}$$

$$=1+\frac{3}{5}\left(1+\frac{1}{5}+\frac{1}{5^2}+\ldots\ldots+\frac{1}{5^{n-2}}\right)-\frac{3n-2}{5^n}$$

$$=1+\frac{3}{5}\cdot\frac{1-\dfrac{1}{5^{n-1}}}{1-\dfrac{1}{5}}-\frac{3n-2}{5^n}$$

$$=1+\frac{3}{4}\left(1-\frac{1}{5^{n-1}}\right)-\frac{3n-2}{5^n}$$

$$=\frac{7}{4}-\frac{12n+7}{4\,.\,5^n};$$

$$\therefore\ S=\frac{35}{16}-\frac{12n+7}{16\,.\,5^{n-1}}.$$

$a\frac{(r^n-1)}{r-1},$

EXAMPLES. V. b.

1. Sum $1+2a+3a^2+4a^3+\ldots$ to n terms.

2. Sum $1+\frac{3}{4}+\frac{7}{16}+\frac{15}{64}+\frac{31}{256}+\ldots$ to infinity.

3. Sum $1+3x+5x^2+7x^3+9x^4+\ldots$ to infinity, x being <1.

4. Sum $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\ldots$ to n terms.

5. Sum $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots$ to infinity.

6. Sum $1+3x+6x^2+10x^3+\ldots$ to infinity, x being <1.

7. Prove that the $(n+1)^{\text{th}}$ term of a G. P., of which the first term is a and the third term b, is equal to the $(2n+1)^{\text{th}}$ term of a G. P. of which the first term is a and the fifth term b.

8. The sum of $2n$ terms of a G. P. whose first term is a and common ratio r is equal to the sum of n of a G. P. whose first term is b and common ratio r^2. Prove that b is equal to the sum of the first two terms of the first series.

9. Find the sum of the infinite series

$$1+(1+b)r+(1+b+b^2)r^2+(1+b+b^2+b^3)r^3+\dots,$$

r and b being proper fractions.

10. The sum of three numbers in G. P. is 70; if the two extremes be multiplied each by 4, and the mean by 5, the products are in A. P.; find the numbers.

11. The first two terms of an infinite G. P. are together equal to 5, and every term is 3 times the sum of all the terms that follow it; find the series.

Sum the following series:

12. $x+a,\ x^2+2a,\ x^3+3a\dots$ to n terms.

13. $x(x+y)+x^2(x^2+y^2)+x^3(x^3+y^3)+\dots$ to n terms.

14. $a+\frac{1}{3},\ 3a-\frac{1}{6},\ 5a+\frac{1}{12}+\dots$ to $2p$ terms.

15. $\frac{2}{3}+\frac{3}{3^2}+\frac{2}{3^3}+\frac{3}{3^4}+\frac{2}{3^5}+\frac{3}{3^6}+\dots$ to infinity.

16. $\frac{4}{7}-\frac{5}{7^2}+\frac{4}{7^3}-\frac{5}{7^4}+\frac{4}{7^5}-\frac{5}{7^6}+\dots$ to infinity.

17. If a, b, c, d be in G. P., prove that

$$(b-c)^2+(c-a)^2+(d-b)^2=(a-d)^2.$$

18. If the arithmetic mean between a and b is twice as great as the geometric mean, shew that $a : b=2+\sqrt{3} : 2-\sqrt{3}$.

19. Find the sum of n terms of the series the r^{th} term of which is

$$(2r+1)\,2^r.$$

20. Find the sum of $2n$ terms of a series of which every even term is a times the term before it, and every odd term c times the term before it, the first term being unity.

21. If S_n denote the sum of n terms of a G. P. whose first term is a, and common ratio r, find the sum of $S_1, S_3, S_5, \dots S_{2n-1}$.

22. If $S_1, S_2, S_3, \dots S_p$ are the sums of infinite geometric series, whose first terms are 1, 2, 3,...p, and whose common ratios are

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \frac{1}{p+1} \text{ respectively,}$$

prove that $$S_1+S_2+S_3+\dots+S_p=\frac{p}{2}(p+3).$$

23. If $r<1$ and positive, and m is a positive integer, shew that

$$(2m+1)r^m(1-r)<1-r^{2m+1}.$$

Hence shew that nr^n is indefinitely small when n is indefinitely great.

ΣS

$\Sigma n = \frac{n(n+1)}{2}$

1, 2. 3. 4

$\Sigma 2n = n(n+1)$

$\Sigma(2n-1) = n^2$

$\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$

$\Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

CHAPTER VI.

HARMONICAL PROGRESSION. THEOREMS CONNECTED WITH THE PROGRESSIONS.

61. DEFINITION. Three quantities a, b, c are said to be in **Harmonical Progression** when $\frac{a}{c} = \frac{a-b}{b-c}$.

Any number of quantities are said to be in Harmonical Progression when every three consecutive terms are in Harmonical Progression.

62. *The reciprocals of quantities in Harmonical Progression are in Arithmetical Progression.*

By definition, if a, b, c are in Harmonical Progression,

$$\frac{a}{c} = \frac{a-b}{b-c};$$

$$\therefore a(b-c) = c(a-b),$$

dividing every term by abc,

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a},$$

which proves the proposition.

63. Harmonical properties are chiefly interesting because of their importance in Geometry and in the Theory of Sound: in Algebra the proposition just proved is the only one of any importance. There is no general formula for the sum of any number of quantities in Harmonical Progression. Questions in H. P. are generally solved by inverting the terms, and making use of the properties of the corresponding A. P.

64. *To find the harmonic mean between two given quantities.*

Let a, b be the two quantities, H their harmonic mean; then $\frac{1}{a}$, $\frac{1}{H}$, $\frac{1}{b}$ are in A. P.;

$$\therefore \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H},$$

$$\frac{2}{H} = \frac{1}{a} + \frac{1}{b},$$

$$H = \frac{2ab}{a+b}.$$

Example. Insert 40 harmonic means between 7 and $\frac{1}{6}$.

Here 6 is the 42^{nd} term of an A. P. whose first term is $\frac{1}{7}$; let d be the common difference; then

$$6 = \frac{1}{7} + 41d; \text{ whence } d = \frac{1}{7}.$$

Thus the arithmetic means are $\frac{2}{7}$, $\frac{3}{7}$, ... $\frac{41}{7}$; and therefore the harmonic means are $3\frac{1}{2}$, $2\frac{1}{3}$, ... $\frac{7}{41}$.

65. If A, G, H be the arithmetic, geometric, and harmonic means between a and b, we have proved

$$A = \frac{a+b}{2} \quad \text{..............................} (1).$$

$$G = \sqrt{ab} \quad \text{..............................} (2).$$

$$H = \frac{2ab}{a+b} \quad \text{..............................} (3).$$

Therefore $$AH = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab = G^2;$$

that is, G is the geometric mean between A and H.

From these results we see that

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \left(\frac{\sqrt{a} - \sqrt{b}}{\sqrt{2}}\right)^2;$$

which is positive if a and b are positive; therefore *the arithmetic mean of any two positive quantities is greater than their geometric mean.*

Also from the equation $G^2 = AH$, we see that G is intermediate in value between A and H; and it has been proved that $A > G$, therefore $G > H$; that is, *the arithmetic, geometric, and harmonic means between any two positive quantities are in descending order of magnitude.*

66. Miscellaneous questions in the Progressions afford scope for skill and ingenuity, the solution being often neatly effected by some special artifice. The student will find the following hints useful.

1. If the same quantity be added to, or subtracted from, all the terms of an A.P., the resulting terms will form an A.P. with the same common difference as before. [Art. 38.]

2. If all the terms of an A.P. be multiplied or divided by the same quantity, the resulting terms will form an A.P., but with a new common difference. [Art. 38.]

3. If all the terms of a G.P. be multiplied or divided by the same quantity, the resulting terms will form a G.P. with the same common ratio as before. [Art. 51.]

4. If $a, b, c, d\ldots$ are in G.P., they are also in *continued proportion*, since, by definition,

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \ldots\ldots = \frac{1}{r}.$$

Conversely, a series of quantities in continued proportion may be represented by $x, xr, xr^2, \ldots\ldots$

Example 1. If a^2, b^2, c^2 are in A.P., shew that $b+c, c+a, a+b$ are in H.P.

By adding $ab+ac+bc$ to each term, we see that

$$a^2+ab+ac+bc, \quad b^2+ba+bc+ac, \quad c^2+ca+cb+ab \text{ are in A.P.};$$

that is $\quad (a+b)(a+c), \ (b+c)(b+a), \ (c+a)(c+b)$ are in A. P.

$\therefore$, dividing each term by $(a+b)(b+c)(c+a)$,

$$\frac{1}{b+c}, \ \frac{1}{c+a}, \ \frac{1}{a+b} \text{ are in A. P.};$$

that is, $\quad b+c, c+a, a+b$ are in H. P.

Example 2. If l the last term, d the common difference, and s the sum of n terms of an A. P. be connected by the equation $8ds=(d+2l)^2$, prove that

$$d=2a.$$

Since the given relation is true for any number of terms, put $n=1$; then

$$a=l=s.$$

Hence by substitution, $\qquad 8ad=(d+2a)^2,$

or $\qquad (d-2a)^2=0;$

$$\therefore\ d=2a.$$

Example 3. If the $p^{\text{th}}, q^{\text{th}}, r^{\text{th}}, s^{\text{th}}$ terms of an A. P. are in G. P., shew that $p-q$, $q-r$, $r-s$ are in G.P.

With the usual notation we have

$$\frac{a+(p-1)\,d}{a+(q-1)\,d}=\frac{a+(q-1)\,d}{a+(r-1)\,d}=\frac{a+(r-1)\,d}{a+(s-1)\,d} \qquad \text{[Art. 66. (4)];}$$

$\therefore$ each of these ratios

$$=\frac{\{a+(p-1)\,d\}-\{a+(q-1)\,d\}}{\{a+(q-1)\,d\}-\{a+(r-1)\,d\}}=\frac{\{a+(q-1)\,d\}-\{a+(r-1)\,d\}}{\{a+(r-1)\,d\}-\{a+(s-1)\,d\}}$$

$$=\frac{p-q}{q-r}=\frac{q-r}{r-s}.$$

Hence $p-q$, $q-r$, $r-s$ are in G.P.

67. The numbers 1, 2, 3,...... are often referred to as the *natural numbers*; the n^{th} term of the series is n, and the sum of the first n terms is $\frac{n}{2}(n+1)$.

68. *To find the sum of the squares of the first* n *natural numbers.*

Let the sum be denoted by S; then

$$S=1^2+2^2+3^2+\ldots\ldots+n^2.$$

We have $\qquad n^3-(n-1)^3=3n^2-3n+1;$

and by changing n into $n-1$,

$$(n-1)^3-(n-2)^3=3(n-1)^2-3(n-1)+1;$$

similarly $\quad (n-2)^3-(n-3)^3=3(n-2)^2-3(n-2)+1;$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$3^3-2^3=3\,.\,3^2-3\,.\,3+1;$$

$$2^3-1^3=3\,.\,2^2-3\,.\,2+1;$$

$$1^3-0^3=3\,.\,1^2-3\,.\,1+1.$$

Hence, by addition,

$$n^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) - 3(1 + 2 + 3 + \ldots + n) + n$$
$$= 3S - \frac{3n(n+1)}{2} + n.$$
$$\therefore\ 3S = n^3 - n + \frac{3n(n+1)}{2}$$
$$= n(n+1)(n - 1 + \tfrac{3}{2});$$
$$\therefore\ S = \frac{n(n+1)(2n+1)}{6}.$$

69. *To find the sum of the cubes of the first* n *natural numbers.*

Let the sum be denoted by S; then

$$S = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3.$$

We have $\qquad n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1;$

$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1;$$
$$(n-2)^4 - (n-3)^4 = 4(n-2)^3 - 6(n-2)^2 + 4(n-2) - 1;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$3^4 - 2^4 = 4 \,.\, 3^3 - 6 \,.\, 3^2 + 4 \,.\, 3 - 1;$$
$$2^4 - 1^4 = 4 \,.\, 2^3 - 6 \,.\, 2^2 + 4 \,.\, 2 - 1;$$
$$1^4 - 0^4 = 4 \,.\, 1^3 - 6 \,.\, 1^2 + 4 \,.\, 1 - 1.$$

Hence, by addition,

$$n^4 = 4S - 6(1^2 + 2^2 + \ldots + n^2) + 4(1 + 2 + \ldots + n) - n;$$
$$\therefore\ 4S = n^4 + n + 6(1^2 + 2^2 + \ldots + n^2) - 4(1 + 2 + \ldots + n)$$
$$= n^4 + n + n(n+1)(2n+1) - 2n(n+1)$$
$$= n(n+1)(n^2 - n + 1 + 2n + 1 - 2)$$
$$= n(n+1)(n^2 + n);$$
$$\therefore\ S = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2.$$

Thus *the sum of the cubes of the first* n *natural numbers is equal to the square of the sum of these numbers.*

The formulæ of this and the two preceding articles may be applied to find the sum of the squares, and the sum of the cubes of the terms of the series

$$a,\ a + d,\ a + 2d, \ldots\ldots\ldots$$

70. In referring to the results we have just proved it will be convenient to introduce a notation which the student will frequently meet with in Higher Mathematics. We shall denote the series

$$1+2+3+\ldots+n \text{ by } \Sigma n;$$
$$1^2+2^2+3^2+\ldots+n^2 \text{ by } \Sigma n^2;$$
$$1^3+2^3+3^3+\ldots+n^3 \text{ by } \Sigma n^3;$$

where Σ placed before a term signifies the sum of all terms of which that term is the general type.

Example 1. Sum the series

$$1\,.\,2+2\,.\,3+3\,.\,4+\ldots \text{ to } n \text{ terms.}$$

The n^{th} term $=n(n+1)=n^2+n$; and by writing down each term in a similar form we shall have two columns, one consisting of the first n natural numbers, and the other of their squares.

$$\therefore \text{ the sum} = \Sigma n^2 + \Sigma n$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\left\{\frac{2n+1}{3}+1\right\}$$
$$= \frac{n(n+1)(n+2)}{3}.$$

Example 2. Sum to n terms the series whose n^{th} term is $2^{n-1}+8n^3-6n^2$.

Let the sum be denoted by S; then

$$S = \Sigma\, 2^{n-1} + 8\Sigma n^3 - 6\Sigma n^2$$
$$= \frac{2^n-1}{2-1} + \frac{8n^2(n+1)^2}{4} - \frac{6n(n+1)(2n+1)}{6}$$
$$= 2^n - 1 + n(n+1)\{2n(n+1)-(2n+1)\}$$
$$= 2^n - 1 + n(n+1)(2n^2-1).$$

EXAMPLES. VI. a.

1. Find the fourth term in each of the following series:

 (1) $2,\ 2\frac{1}{2},\ 3\frac{1}{3},\ldots$

 (2) $2,\ 2\frac{1}{2},\ 3,\ldots$

 (3) $2,\ 2\frac{1}{2},\ 3\frac{1}{8},\ldots$

2. Insert two harmonic means between 5 and 11.

3. Insert four harmonic means between $\frac{2}{3}$ and $\frac{2}{13}$.

4. If 12 and $9\frac{3}{5}$ are the geometric and harmonic means, respectively, between two numbers, find them.

5. If the harmonic mean between two quantities is to their geometric means as 12 to 13, prove that the quantities are in the ratio of 4 to 9.

6. If a, b, c be in H. P., shew that
$$a : a-b = a+c : a-c.$$

7. If the m^{th} term of a H. P. be equal to n, and the n^{th} term be equal to m, prove that the $(m+n)^{\text{th}}$ term is equal to $\dfrac{mn}{m+n}$.

8. If the p^{th}, q^{th}, r^{th} terms of a H. P. be a, b, c respectively, prove that
$$(q-r)\,bc+(r-p)\,ca+(p-q)\,ab=0.$$

9. If b is the harmonic mean between a and c, prove that
$$\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}.$$

Find the sum of n terms of the series whose n^{th} term is

10. $3n^2-n$.   11. $n^3+\frac{3}{2}n$.   12. $n(n+2)$.

13. $n^2(2n+3)$.   14. 3^n-2^n.   15. $3(4^n+2n^2)-4n^3$.

16. If the $(m+1)^{\text{th}}$, $(n+1)^{\text{th}}$, and $(r+1)^{\text{th}}$ terms of an A. P. are in G. P., and m, n, r are in H. P., shew that the ratio of the common difference to the first term in the A. P. is $-\dfrac{2}{n}$.

17. If l, m, n are three numbers in G. P., prove that the first term of an A. P. whose l^{th}, m^{th}, and n^{th} terms are in H. P. is to the common difference as $m+1$ to 1.

18. If the sum of n terms of a series be $a+bn+cn^2$, find the n^{th} term and the nature of the series.

19. Find the sum of n terms of the series whose n^{th} term is
$$4n(n^2+1)-(6n^2+1).$$

20. If between any two quantities there be inserted two arithmetic means A_1, A_2; two geometric means G_1, G_2; and two harmonic means H_1, H_2; shew that $G_1G_2 : H_1H_2 = A_1+A_2 : H_1+H_2$.

21. If p be the first of n arithmetic means between two numbers, and q the first of n harmonic means between the same two numbers, prove that the value of q cannot lie between p and $\left(\dfrac{n+1}{n-1}\right)^2 p$.

22. Find the sum of the cubes of the terms of an A. P., and shew that it is exactly divisible by the sum of the terms.

Piles of Shot and Shells.

71. *To find the number of shot arranged in a complete pyramid on a square base.*

Suppose that each side of the base contains n shot; then the number of shot in the lowest layer is n^2; in the next it is $(n-1)^2$; in the next $(n-2)^2$; and so on, up to a single shot at the top.

$$\therefore\ S = n^2 + (n-1)^2 + (n-2)^2 + \dots + 1$$

$$= \frac{n(n+1)(2n+1)}{6} \qquad \text{[Art. 68.]}$$

72. *To find the number of shot arranged in a complete pyramid the base of which is an equilateral triangle.*

Suppose that each side of the base contains n shot; then the number of shot in the lowest layer is

$$n + (n-1) + (n-2) + \dots\dots + 1;$$

that is,
$$\frac{n(n+1)}{2} \text{ or } \frac{1}{2}(n^2+n).$$

In this result write $n-1$, $n-2$,...... for n, and we thus obtain the number of shot in the 2nd, 3rd,...... layers.

$$\therefore\ S = \tfrac{1}{2}(\Sigma n^2 + \Sigma n)$$

$$= \frac{n(n+1)(n+2)}{6} \qquad \text{[Art. 70.]}$$

73. *To find the number of shot arranged in a complete pyramid the base of which is a rectangle.*

Let m and n be the number of shot in the long and short side respectively of the base.

The top layer consists of a single row of $m-(n-1)$, or $m-n+1$ shot;

in the next layer the number is $2(m-n+2)$;

in the next layer the number is $3(m-n+3)$;

and so on;

in the lowest layer the number is $n(m-n+n)$.

$$\therefore\ S = (m-n+1) + 2(m-n+2) + 3(m-n+3) + \ldots + n(m-n+n)$$

$$= (m-n)(1+2+3+\ldots+n) + (1^2+2^2+3^2+\ldots+n^2)$$

$$= \frac{(m-n)\,n\,(n+1)}{2} + \frac{n\,(n+1)\,(2n+1)}{6}$$

$$= \frac{n\,(n+1)}{6}\{3\,(m-n)+2n+1\}$$

$$= \frac{n\,(n+1)\,(3m-n+1)}{6}.$$

74. *To find the number of shot arranged in an incomplete pyramid the base of which is a rectangle.*

Let a and b denote the number of shot in the two sides of the top layer, n the number of layers.

In the top layer the number of shot is ab;

in the next layer the number is $(a+1)(b+1)$;

in the next layer the number is $(a+2)(b+2)$;

and so on;

in the lowest layer the number is $(a+\overline{n-1})(b+\overline{n-1})$

or $$ab+(a+b)(n-1)+(n-1)^2.$$

$$\therefore\ S = abn + (a+b)\,\Sigma\,(n-1) + \Sigma\,(n-1)^2$$

$$= abn + \frac{(n-1)\,n\,(a+b)}{2} + \frac{(n-1)\,n\,(2\,.\,\overline{n-1}+1)}{6}$$

$$= \frac{n}{6}\{6ab + 3\,(a+b)(n-1) + (n-1)(2n-1)\}.$$

75. In numerical examples it is generally easier to use the following method.

Example. Find the number of shot in an incomplete square pile of 16 courses, having 12 shot in each side of the top.

If we place on the given pile a square pile having 11 shot in each side of the base, we obtain a complete square pile of 27 courses;

and number of shot in the complete pile $= \dfrac{27\times 28\times 55}{6} = 6930$; [Art 71.]

also number of shot in the added pile $= \dfrac{11\times 12\times 23}{6} = 506$;

$\therefore$ number of shot in the incomplete pile $= 6424$.

EXAMPLES. VI. b.

Find the number of shot in

1. A square pile, having 15 shot in each side of the base.

2. A triangular pile, having 18 shot in each side of the base.

3. A rectangular pile, the length and the breadth of the base containing 50 and 28 shot respectively.

4. An incomplete triangular pile, a side of the base having 25 shot, and a side of the top 14.

5. An incomplete square pile of 27 courses, having 40 shot in each side of the base.

6. The number of shot in a complete rectangular pile is 24395; if there are 34 shot in the breadth of the base, how many are there in its length?

7. The number of shot in the top layer of a square pile is 169, and in the lowest layer is 1089; how many shot does the pile contain?

8. Find the number of shot in a complete rectangular pile of 15 courses, having 20 shot in the longer side of its base.

9. Find the number of shot in an incomplete rectangular pile, the number of shot in the sides of its upper course being 11 and 18, and the number in the shorter side of its lowest course being 30.

10. What is the number of shot required to complete a rectangular pile having 15 and 6 shot in the longer and shorter side, respectively, of its upper course?

11. The number of shot in a triangular pile is greater by 150 than half the number of shot in a square pile, the number of layers in each being the same; find the number of shot in the lowest layer of the triangular pile.

12. Find the number of shot in an incomplete square pile of 16 courses when the number of shot in the upper course is 1005 less than in the lowest course.

13. Shew that the number of shot in a square pile is one-fourth the number of shot in a triangular pile of double the number of courses.

14. If the number of shot in a triangular pile is to the number of shot in a square pile of double the number of courses as 13 to 175; find the number of shot in each pile.

15. The value of a triangular pile of 16 lb. shot is £51; if the value of iron be 10*s.* 6*d.* per cwt., find the number of shot in the lowest layer.

16. If from a complete square pile of n courses a triangular pile of the same number of courses be formed; shew that the remaining shot will be just sufficient to form another triangular pile, and find the number of shot in its side.

CHAPTER VII.

SCALES OF NOTATION.

76. The ordinary numbers with which we are acquainted in Arithmetic are expressed by means of multiples of powers of 10; for instance

$$25 = 2 \times 10 + 5;$$

$$4705 = 4 \times 10^3 + 7 \times 10^2 + 0 \times 10 + 5.$$

This method of representing numbers is called the **common** or **denary scale of notation,** and ten is said to be the **radix** of the scale. The symbols employed in this system of notation are the nine digits and zero.

In like manner any number other than ten may be taken as the radix of a scale of notation; thus if 7 is the radix, a number expressed by 2453 represents $2 \times 7^3 + 4 \times 7^2 + 5 \times 7 + 3$; and in this scale no digit higher than 6 can occur.

Again in a scale whose radix is denoted by r the above number 2453 stands for $2r^3 + 4r^2 + 5r + 3$. More generally, if in the scale whose radix is r we denote the digits, beginning with that in the units' place, by $a_0, a_1, a_2, \ldots a_n$; then the number so formed will be represented by

$$a_n r^n + a_{n-1} r^{n-1} + a_{n-2} r^{n-2} + \ldots + a_2 r^2 + a_1 r + a_0,$$

where the coefficients $a_n, a_{n-1}, \ldots a_0$ are integers, all less than r, of which any one or more after the first may be zero.

Hence in this scale the digits are r in number, their values ranging from 0 to $r - 1$.

77. The names Binary, Ternary, Quaternary, Quinary, Senary, Septenary, Octenary, Nonary, Denary, Undenary, and Duodenary are used to denote the scales corresponding to the values *two*, *three*,...*twelve* of the radix.

In the undenary, duodenary,... scales we shall require symbols to represent the digits which are greater than nine. It is unusual to consider any scale higher than that with radix twelve; when necessary we shall employ the symbols t, e, T as digits to denote 'ten', 'eleven' and 'twelve'.

It is especially worthy of notice that in every scale 10 is the symbol not for 'ten', but for the radix itself.

78. The ordinary operations of Arithmetic may be performed in any scale; but, bearing in mind that the successive powers of the radix are no longer powers of ten, in determining the *carrying figures* we must not divide by ten, but by the radix of the scale in question.

Example 1. In the scale of eight subtract 371532 from 530225, and multiply the difference by 27.

$$\begin{array}{r} 530225 \\ 371532 \\ \hline 136473 \end{array} \qquad\qquad \begin{array}{r} 136473 \\ 27 \\ \hline 1226235 \\ 275166 \\ \hline 4200115 \end{array}$$

Explanation. After the first figure of the subtraction, since we cannot take 3 from 2 we add 8; thus we have to take 3 from ten, which leaves 7; then 6 from ten, which leaves 4; then 2 from eight which leaves 6; and so on.

Again, in multiplying by 7, we have

$$3 \times 7 = \text{twenty one} = 2 \times 8 + 5;$$

we therefore put down 5 and carry 2.

Next $\qquad 7 \times 7 + 2 = \text{fifty one} = 6 \times 8 + 3;$

put down 3 and carry 6; and so on, until the multiplication is completed.

In the addition,

$$3 + 6 = \text{nine} = 1 \times 8 + 1;$$

we therefore put down 1 and carry 1.

Similarly $\qquad 2 + 6 + 1 = \text{nine} = 1 \times 8 + 1;$

and $\qquad 6 + 1 + 1 = \text{eight} = 1 \times 8 + 0;$

and so on.

Example 2. Divide $15et20$ by 9 in the scale of twelve.

$$\begin{array}{l} 9)\underline{15et20} \\ \quad\; 1ee96 \ldots 6. \end{array}$$

Explanation. Since $15 = 1 \times T + 5 = \text{seventeen} = 1 \times 9 + 8$, we put down 1 and carry 8.

Also $8 \times T + e = \text{one hundred and seven} = e \times 9 + 8$; we therefore put down e and carry 8; and so on.

Example 3. Find the square root of 442641 in the scale of seven.

```
         442641(546
         34
    134 | 1026
        |  602
   1416 | 12441
        | 12441
```

EXAMPLES. VII. a.

1. Add together 23241, 4032, 300421 in the scale of five.

2. Find the sum of the nonary numbers 303478, 150732, 264305.

3. Subtract 1732765 from 3673124 in the scale of eight.

4. From 3*te*756 take 2*e*46*t*2 in the duodenary scale.

5. Divide the difference between 1131315 and 235143 by 4 in the scale of six.

6. Multiply 6431 by 35 in the scale of seven.

7. Find the product of the nonary numbers 4685, 3483.

8. Divide 102432 by 36 in the scale of seven.

9. In the ternary scale subtract 121012 from 11022201, and divide the result by 1201.

10. Find the square root of 300114 in the quinary scale.

11. Find the square of *tttt* in the scale of eleven.

12. Find the G. C. M. of 2541 and 3102 in the scale of seven.

13. Divide 14332216 by 6541 in the septenary scale.

14. Subtract 20404020 from 103050301 and find the square root of the result in the octenary scale.

15. Find the square root of *eet*001 in the scale of twelve.

16. The following numbers are in the scale of six, find by the ordinary rules, without transforming to the denary scale:

(1) the G. C. M. of 31141 and 3102;

(2) the L. C. M. of 23, 24, 30, 32, 40, 41, 43, 50.

79. *To express a given integral number in any proposed scale.*

Let N be the given number, and r the radix of the proposed scale.

Let $a_0, a_1, a_2, \ldots a_n$ be the required digits by which N is to be expressed, beginning with that in the units' place; then

$$N = a_n r^n + a_{n-1} r^{n-1} + \ldots + a_2 r^2 + a_1 r + a_0.$$

We have now to find the values of $a_0, a_1, a_2, \ldots a_n$

Divide N by r, then the remainder is a_0, and the quotient is

$$a_n r^{n-1} + a_{n-1} r^{n-2} + \ldots + a_2 r + a_1.$$

If this quotient is divided by r, the remainder is a_1;
if the next quotient .... a_2;
and so on, until there is no further quotient.

Thus all the required digits $a_0, a_1, a_2, \ldots a_n$ are determined by successive divisions by the radix of the proposed scale.

Example 1. Express the denary number 5213 in the scale of seven.

$$\begin{array}{r l}
7)\underline{5213} & \\
7)\underline{744} & \ldots\ldots 5 \\
7)\underline{106} & \ldots\ldots 2 \\
7)\underline{15} & \ldots\ldots 1 \\
2 & \ldots\ldots 1
\end{array}$$

Thus $\qquad 5213 = 2 \times 7^4 + 1 \times 7^3 + 1 \times 7^2 + 2 \times 7 + 5;$

and the number required is 21125.

Example 2. Transform 21125 from scale seven to scale eleven.

$$\begin{array}{r l}
e)\underline{21125} & \\
e)\underline{1244} & \ldots\ldots t \\
e)\underline{61} & \ldots\ldots 0 \\
3 & \ldots\ldots t
\end{array}$$

$\therefore$ the required number is $3t0t$.

Explanation. In the first line of work

$$21 = 2 \times 7 + 1 = \text{fifteen} = 1 \times e + 4;$$

therefore on dividing by e we put down 1 and carry 4.

Next $4 \times 7 + 1 = \text{twenty nine} = 2 \times e + 7$;
therefore we put down 2 and carry 7; and so on.

Example 3. Reduce 7215 from scale twelve to scale ten by working in scale ten, and verify the result by working in the scale twelve.

$$\text{In scale of ten}\left\{\begin{array}{r}
7215 \\
\underline{12} \\
86 \\
\underline{12} \\
1033 \\
\underline{12} \\
12401
\end{array}\right. \qquad\qquad \left.\begin{array}{r l}
t)\underline{7215} & \\
t)\underline{874} & \ldots\ldots 1 \\
t)\underline{t4} & \ldots\ldots 0 \\
t)\underline{10} & \ldots\ldots 4 \\
1 & \ldots\ldots 2
\end{array}\right\}\text{In scale of twelve}$$

Thus the result is 12401 in each case.

Explanation. 7215 in scale twelve means $7 \times 12^3 + 2 \times 12^2 + 1 \times 12 + 5$ in scale ten. The calculation is most readily effected by writing this expression in the form $[\{(7 \times 12 + 2)\} \times 12 + 1] \times 12 + 5$; thus we multiply 7 by 12, and add 2 to the product; then we multiply 86 by 12 and add 1 to the product; then 1033 by 12 and add 5 to the product.

80. Hitherto we have only discussed whole numbers; but fractions may also be expressed in any scale of notation; thus

$$\cdot 25 \text{ in scale ten denotes } \frac{2}{10} + \frac{5}{10^2};$$

$$\cdot 25 \text{ in scale six denotes } \frac{2}{6} + \frac{5}{6^2};$$

$$\cdot 25 \text{ in scale } r \text{ denotes } \frac{2}{r} + \frac{5}{r^2}.$$

Fractions thus expressed in a form analogous to that of ordinary decimal fractions are called **radix-fractions**, and the point is called the **radix-point**. The general type of such fractions in scale r is

$$\frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \dots\dots;$$

where $b_1, b_2, b_3, \dots$ are integers, all less than r, of which any one or more may be zero.

81. *To express a given radix fraction in any proposed scale.*

Let F be the given fraction, and r the radix of the proposed scale.

Let $b_1, b_2, b_3, \dots$ be the required digits beginning from the left; then

$$F = \frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \dots\dots$$

We have now to find the values of $b_1, b_2, b_3, \dots\dots$

Multiply both sides of the equation by r; then

$$rF = b_1 + \frac{b_2}{r} + \frac{b_3}{r^2} + \dots\dots;$$

Hence b_1 is equal to the integral part of rF; and, if we denote the fractional part by F_1, we have

$$F_1 = \frac{b_2}{r} + \frac{b_3}{r^2} + \dots\dots$$

Multiply again by r; then, as before, b_2 is the integral part of rF_1; and similarly by successive multiplications by r, each of the digits may be found, and the fraction expressed in the proposed scale.

If in the successive multiplications by r any one of the products is an integer the process terminates at this stage, and the given fraction can be expressed by a finite number of digits. But if none of the products is an integer the process will never terminate, and in this case the digits recur, forming a radix-fraction analogous to a recurring decimal.

Example 1. Express $\frac{13}{16}$ as a radix fraction in scale six.

$$\frac{13}{16}\times 6=\frac{13\times 3}{8}=4+\frac{7}{8};$$
$$\frac{7}{8}\times 6=\frac{7\times 3}{4}=5+\frac{1}{4};$$
$$\frac{1}{4}\times 6=\frac{1\times 3}{2}=1+\frac{1}{2};$$
$$\frac{1}{2}\times 6=3.$$

$$\therefore \text{ the required fraction}=\frac{4}{6}+\frac{5}{6^2}+\frac{1}{6^3}+\frac{3}{6^4}$$
$$=\cdot 4513.$$

Example 2. Transform $16064\cdot 24$ from scale eight to scale five.

We must treat the integral and the fractional parts separately,

$$\begin{array}{r|l} 5)\underline{16064} & \\ 5)\underline{2644} & 0 \\ 5)\underline{440} & 4 \\ 5)\underline{71} & 3 \\ 5)\underline{13} & 2 \\ 2 & 1 \end{array} \qquad\qquad \begin{array}{r} \cdot 24 \\ \underline{5} \\ 1\cdot 44 \\ \underline{5} \\ 2\cdot 64 \\ \underline{5} \\ 4\cdot 04 \\ \underline{5} \\ 0\cdot 24 \end{array}$$

After this the digits in the fractional part recur; hence the required number is $212340\cdot\dot{1}24\dot{0}$.

82. *In any scale of notation of which the radix is r, the sum of the digits of any whole number divided by r − 1 will leave the same remainder as the whole number divided by r − 1.*

Let N denote the number, $a_0, a_1, a_2, \ldots\ldots a_n$ the digits beginning with that in the units' place, and S the sum of the digits; then

$$N=a_0+a_1r+a_2r^2+\ldots\ldots+a_{n-1}r^{n-1}+a_nr^n;$$
$$S=a_0+a_1+a_2+\ldots\ldots+a_{n-1}+a_n$$
$$\therefore N-S=a_1(r-1)+a_2(r^2-1)+\ldots\ldots+a_{n-1}(r^{n-1}-1)+a_n(r^n-1).$$

Now every term on the right hand side is divisible by $r-1$;

$$\therefore \frac{N-S}{r-1} = \text{an integer};$$

that is,
$$\frac{N}{r-1} = I + \frac{S}{r-1},$$

where I is some integer; which proves the proposition.

Hence a number in scale r will be divisible by $r-1$ when the sum of its digits is divisible by $r-1$.

83. By taking $r=10$ we learn from the above proposition that a number divided by 9 will leave the same remainder as the sum of its digits divided by 9. The rule known as "casting out the nines" for testing the accuracy of multiplication is founded on this property.

The rule may be thus explained:

Let two numbers be represented by $9a+b$ and $9c+d$, and their product by P; then

$$P = 81ac + 9bc + 9ad + bd.$$

Hence $\frac{P}{9}$ has the same remainder as $\frac{bd}{9}$; and therefore the *sum of the digits* of P, when divided by 9, gives the same remainder as the *sum of the digits* of bd, when divided by 9. If on trial this should not be the case, the multiplication must have been incorrectly performed. In practice b and d are readily found from the sums of the digits of the two numbers to be multiplied together.

Example. Can the product of 31256 and 8427 be 263395312?

The sums of the digits of the multiplicand, multiplier, and product are 17, 21, and 34 respectively; again, the sums of the digits of these three numbers are 8, 3, and 7, whence $bd = 8 \times 3 = 24$, which has 6 for the sum of the digits; thus we have two different remainders, 6 and 7, and the multiplication is incorrect.

84. *If* N *denote any number in the scale of* r, *and* D *denote the difference, supposed positive, between the sums of the digits in the odd and the even places; then* $N - D$ *or* $N + D$ *is a multiple of* $r + 1$.

Let $a_0, a_1, a_2, \ldots\ldots a_n$ denote the digits beginning with that in the units' place; then

$$N = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \ldots\ldots + a_{n-1} r^{n-1} + a_n r^n.$$

$\therefore\ N - a_0 + a_1 - a_2 + a_3 - \ldots = a_1(r+1) + a_2(r^2-1) + a_3(r^3+1) + \ldots;$ and the last term on the right will be $a_n(r^n+1)$ or $a_n(r^n-1)$ according as n is odd or even. Thus every term on the right is divisible by $r+1$; hence

$$\frac{N - (a_0 - a_1 + a_2 - a_3 + \ldots\ldots)}{r+1} = \text{an integer}.$$

Now $$a_0 - a_1 + a_2 - a_3 + \ldots\ldots = \pm D;$$

$$\therefore\ \frac{N \mp D}{r+1} \text{ is an integer};$$

which proves the proposition.

COR. If the sum of the digits in the even places is equal to the sum of the digits in the odd places, $D = 0$, and N is divisible by $r+1$.

Example 1. Prove that 4·41 is a square number in any scale of notation whose radix is greater than 4.

Let r be the radix; then

$$4{\cdot}41 = 4 + \frac{4}{r} + \frac{1}{r^2} = \left(2 + \frac{1}{r}\right)^2;$$

thus the given number is the square of 2·1.

Example 2. In what scale is the denary number 2·4375 represented by 2·13?

Let r be the scale; then

$$2 + \frac{1}{r} + \frac{3}{r^2} = 2{\cdot}4375 = 2\frac{7}{16};$$

whence $$7r^2 - 16r - 48 = 0;$$

that is, $$(7r+12)(r-4) = 0.$$

Hence the radix is 4.

Sometimes it is best to use the following method.

Example 3. In what scale will the nonary number 25607 be expressed by 101215?

The required scale must be less than 9, since the new number *appears* the greater; also it must be greater than 5; therefore the required scale must be 6, 7, or 8; and *by trial* we find that it is 7.

Example 4. By working in the duodenary scale, find the height of a rectangular solid whose volume is 364 cub. ft. 1048 cub. in., and the area of whose base is 46 sq. ft. 8 sq. in.

The volume is $364\frac{1048}{1728}$ cub. ft., which expressed in the scale of twelve is 264·734 cub. ft.

The area is $46\frac{8}{144}$ sq. ft., which expressed in the scale of twelve is $3t$·08. We have therefore to divide 264·734 by $3t$·08 in the scale of twelve.

$$\begin{array}{r} 3t08)26473{\cdot}4(7{\cdot}e \\ \underline{22t48} \\ 36274 \\ \underline{36274} \end{array}$$

Thus the height is 7ft. 11in.

EXAMPLES. VII. b.

1. Express 4954 in the scale of seven.
2. Express 624 in the scale of five.
3. Express 206 in the binary scale.
4. Express 1458 in the scale of three.
5. Express 5381 in powers of nine.
6. Transform 212231 from scale four to scale five.
7. Express the duodenary number 398*e* in powers of 10.
8. Transform 6*t*12 from scale twelve to scale eleven.
9. Transform 213014 from the senary to the nonary scale.
10. Transform 23861 from scale nine to scale eight.
11. Transform 400803 from the nonary to the quinary scale.
12. Express the septenary number 20665152 in powers of 12.
13. Transform *ttteee* from scale twelve to the common scale.
14. Express $\frac{3}{10}$ as a radix fraction in the septenary scale.
15. Transform 17·15625 from scale ten to scale twelve.
16. Transform 200·211 from the ternary to the nonary scale.
17. Transform 71·03 from the duodenary to the octenary scale.
18. Express the septenary fraction $\frac{1552}{2626}$ as a denary vulgar fraction in its lowest terms.
19. Find the denary value of the septenary numbers $\cdot\dot{4}$ and $\cdot\dot{4}\dot{2}$.
20. In what scale is the denary number 182 denoted by 222?
21. In what scale is the denary fraction $\frac{25}{128}$ denoted by ·0302?

+3 - 2x2

22. Find the radix of the scale in which 554 represents the square of 24.

23. In what scale is 511197 denoted by 1746335?

24. Find the radix of the scale in which the numbers denoted by 479, 698, 907 are in arithmetical progression.

25. In what scale are the radix-fractions ·16, ·20, ·28 in geometric progression?

26. The number 212542 is in the scale of six; in what scale will it be denoted by 17486?

27. Shew that 148·84 is a perfect square in every scale in which the radix is greater than eight.

28. Shew that 1234321 is a perfect square in any scale whose radix is greater than 4; and that the square root is always expressed by the same four digits.

29. Prove that 1·331 is a perfect cube in any scale whose radix is greater than three.

30. Find which of the weights 1, 2, 4, 8, 16,... lbs. must be used to weigh one ton.

31. Find which of the weights 1, 3, 9, 27, 81,... lbs. must be used to weigh ten thousand lbs., not more than one of each kind being used but in either scale that is necessary.

32. Shew that 1367631 is a perfect cube in every scale in which the radix is greater than seven.

33. Prove that in the ordinary scale a number will be divisible by 8 if the number formed by its last three digits is divisible by eight.

34. Prove that the square of $rrrr$ in the scale of s is $rrrq0001$, where q, r, s are any three consecutive integers.

35. If any number N be taken in the scale r, and a new number N' be formed by altering the order of its digits in any way, shew that the difference between N and N' is divisible by $r-1$.

36. If a number has an even number of digits, shew that it is divisible by $r+1$ if the digits equidistant from each end are the same.

37. If in the ordinary scale S_1 be the sum of the digits of a number N, and $3S_2$ be the sum of the digits of the number $3N$, prove that the difference between S_1 and S_2 is a multiple of 3.

38. Shew that in the ordinary scale any number formed by writing down three digits and then repeating them in the same order is a multiple of 7, 11, and 13.

39. In a scale whose radix is odd, shew that the sum of the digits of any number will be odd if the number be odd, and even if the number be even.

40. If n be odd, and a number in the denary scale be formed by writing down n digits and then repeating them in the same order, shew that it will be divisible by the number formed by the n digits, and also by 9090...9091 containing $n-1$ digits.

CHAPTER VIII.

SURDS AND IMAGINARY QUANTITIES.

85. In the *Elementary Algebra*, Art. 272, it is proved that the denominator of any expression of the form $\dfrac{a}{\sqrt{b}+\sqrt{c}}$ can be rationalised by multiplying the numerator and the denominator by $\sqrt{b}-\sqrt{c}$, the surd *conjugate* to the denominator.

Similarly, in the case of a fraction of the form $\dfrac{a}{\sqrt{b}+\sqrt{c}+\sqrt{d}}$, where the denominator involves three quadratic surds, we may by two operations render that denominator rational.

For, first multiply both numerator and denominator by $\sqrt{b}+\sqrt{c}-\sqrt{d}$; the denominator becomes $(\sqrt{b}+\sqrt{c})^2-(\sqrt{d})^2$ or $b+c-d+2\sqrt{bc}$. Then multiply both numerator and denominator by $(b+c-d)-2\sqrt{bc}$; the denominator becomes $(b+c-d)^2-4bc$, which is a rational quantity.

Example. Simplify $\dfrac{12}{3+\sqrt{5}-2\sqrt{2}}$.

The expression
$$=\frac{12\,(3+\sqrt{5}+2\sqrt{2})}{(3+\sqrt{5})^2-(2\sqrt{2})^2}$$
$$=\frac{12\,(3+\sqrt{5}+2\sqrt{2})}{6+6\sqrt{5}}$$
$$=\frac{2\,(3+\sqrt{5}+2\sqrt{2})\,(\sqrt{5}-1)}{(\sqrt{5}+1)\,(\sqrt{5}-1)}$$
$$=\frac{2+2\sqrt{5}+2\sqrt{10}-2\sqrt{2}}{2}$$
$$=1+\sqrt{5}+\sqrt{10}-\sqrt{2}$$

86. *To find the factor which will rationalise any given binomial surd.*

Case I. Suppose the given surd is $\sqrt[p]{a} - \sqrt[q]{b}$.

Let $\sqrt[p]{a} = x$, $\sqrt[q]{b} = y$, and let n be the L.C.M. of p and q; then x^n and y^n are both rational.

Now $x^n - y^n$ is divisible by $x - y$ for all values of n, and

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots\dots + y^{n-1}).$$

Thus the rationalising factor is

$$x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots\dots + y^{n-1};$$

and the rational product is $x^n - y^n$.

Case II. Suppose the given surd is $\sqrt[p]{a} + \sqrt[q]{b}$.

Let x, y, n have the same meanings as before; then

(1) If n is even, $x^n - y^n$ is divisible by $x + y$, and

$$x^n - y^n = (x + y)(x^{n-1} - x^{n-2}y + \dots\dots + xy^{n-2} - y^{n-1}).$$

Thus the rationalising factor is

$$x^{n-1} - x^{n-2}y + \dots\dots + xy^{n-2} - y^{n-1};$$

and the rational product is $x^n - y^n$.

(2) If n is odd, $x^n + y^n$ is divisible by $x + y$, and

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \dots\dots - xy^{n-2} + y^{n-1}).$$

Thus the rationalising factor is

$$x^{n-1} - x^{n-2}y + \dots\dots - xy^{n-2} + y^{n-1};$$

and the rational product is $x^n + y^n$.

Example 1. Find the factor which will rationalise $\sqrt{3} + \sqrt[3]{5}$.

Let $x = 3^{\frac{1}{2}}$, $y = 5^{\frac{1}{3}}$; then x^6 and y^6 are both rational, and

$$x^6 - y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5);$$

thus, substituting for x and y, the required factor is

$$3^{\frac{5}{2}} - 3^{\frac{4}{2}} \cdot 5^{\frac{1}{3}} + 3^{\frac{3}{2}} \cdot 5^{\frac{2}{3}} - 3^{\frac{2}{2}} \cdot 5^{\frac{3}{3}} + 3^{\frac{1}{2}} \cdot 5^{\frac{4}{3}} - 5^{\frac{5}{3}},$$

or

$$3^{\frac{5}{2}} - 9 \cdot 5^{\frac{1}{3}} + 3^{\frac{3}{2}} \cdot 5^{\frac{2}{3}} - 15 + 3^{\frac{1}{2}} \cdot 5^{\frac{4}{3}} - 5^{\frac{5}{3}};$$

and the rational product is $3^{\frac{6}{2}} - 5^{\frac{6}{3}} = 3^3 - 5^2 = 2$.

Example 2. Express $\left(5^{\frac{1}{2}}+9^{\frac{1}{8}}\right) \div \left(5^{\frac{1}{2}}-9^{\frac{1}{8}}\right)$
as an equivalent fraction with a rational denominator.

To rationalise the denominator, which is equal to $5^{\frac{1}{2}}-3^{\frac{1}{4}}$, put $5^{\frac{1}{2}}=x$, $3^{\frac{1}{4}}=y$; then since $x^4-y^4=(x-y)(x^3+x^2y+xy^2+y^3)$

the required factor is $\quad 5^{\frac{3}{2}}+5^{\frac{2}{2}}.3^{\frac{1}{4}}+5^{\frac{1}{2}}.3^{\frac{2}{4}}+3^{\frac{3}{4}}$;

and the rational denominator is $5^{\frac{4}{2}}-3^{\frac{4}{4}}=5^2-3=22$.

$$\therefore \text{ the expression } = \frac{\left(5^{\frac{1}{2}}+3^{\frac{1}{4}}\right)\left(5^{\frac{3}{2}}+5^{\frac{2}{2}}.3^{\frac{1}{4}}+5^{\frac{1}{2}}.3^{\frac{2}{4}}+3^{\frac{3}{4}}\right)}{22}$$

$$=\frac{5^{\frac{4}{2}}+2.5^{\frac{3}{2}}.3^{\frac{1}{4}}+2.5^{\frac{2}{2}}.3^{\frac{2}{4}}+2.5^{\frac{1}{2}}.3^{\frac{3}{4}}+3^{\frac{4}{4}}}{22}$$

$$=\frac{14+5^{\frac{3}{2}}.3^{\frac{1}{4}}+5.3^{\frac{1}{2}}+5^{\frac{1}{2}}.3^{\frac{3}{4}}}{11}$$

87. We have shewn in the *Elementary Algebra*, Art. 277, how to find the square root of a binomial quadratic surd. We may sometimes extract the square root of an expression containing more than two quadratic surds, such as $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$.

Assume $\quad \sqrt{a+\sqrt{b}+\sqrt{c}+\sqrt{d}}=\sqrt{x}+\sqrt{y}+\sqrt{z}$;

$$\therefore a+\sqrt{b}+\sqrt{c}+\sqrt{d}=x+y+z+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{yz}.$$

If then $\quad 2\sqrt{xy}=\sqrt{b},\ 2\sqrt{xz}=\sqrt{c},\ 2\sqrt{yz}=\sqrt{d}$,

and if, at the same time, the values of x, y, z thus found satisfy $x+y+z=a$, we shall have obtained the required root.

Example. Find the square root of $21-4\sqrt{5}+8\sqrt{3}-4\sqrt{15}$.

Assume $\quad \sqrt{21-4\sqrt{5}+8\sqrt{3}-4\sqrt{15}}=\sqrt{x}+\sqrt{y}-\sqrt{z}$;

$$\therefore 21-4\sqrt{5}+8\sqrt{3}-4\sqrt{15}=x+y+z+2\sqrt{xy}-2\sqrt{xz}-2\sqrt{yz}.$$

Put $\quad 2\sqrt{xy}=8\sqrt{3},\ 2\sqrt{xz}=4\sqrt{15},\ 2\sqrt{yz}=4\sqrt{5}$;

by multiplication, $\quad xyz=240$; that is $\sqrt{xyz}=4\sqrt{15}$;
whence it follows that $\quad \sqrt{x}=2\sqrt{3},\ \sqrt{y}=2,\ \sqrt{z}=\sqrt{5}$.

And since these values satisfy the equation $x+y+z=21$, the required root is $2\sqrt{3}+2-\sqrt{5}$.

88. *If* $\sqrt[3]{a+\sqrt{b}} = x + \sqrt{y}$, *then will* $\sqrt[3]{a-\sqrt{b}} = x - \sqrt{y}$.

For, by cubing, we obtain

$$a + \sqrt{b} = x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y}.$$

Equating rational and irrational parts, we have

$$a = x^3 + 3xy, \quad \sqrt{b} = 3x^2\sqrt{y} + y\sqrt{y};$$

$$\therefore \; a - \sqrt{b} = x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y};$$

that is,
$$\sqrt[3]{a-\sqrt{b}} = x - \sqrt{y}.$$

Similarly, by the help of the Binomial Theorem, Chap. XIII., it may be proved that if

$$\sqrt[n]{a+\sqrt{b}} = x + \sqrt{y}, \text{ then } \sqrt[n]{a-\sqrt{b}} = x - \sqrt{y},$$

where n is any positive integer.

89. By the following method the cube root of an expression of the form $a \pm \sqrt{b}$ may sometimes be found.

Suppose
$$\sqrt[3]{a+\sqrt{b}} = x + \sqrt{y};$$

then
$$\sqrt[3]{a-\sqrt{b}} = x - \sqrt{y}.$$

$$\therefore \; \sqrt[3]{a^2-b} = x^2 - y \quad \text{.......................(1)}.$$

Again, as in the last article,

$$a = x^3 + 3xy. \quad \text{...........................(2)}.$$

The values of x and y have to be determined from (1) and (2).

In (1) suppose that $\sqrt[3]{a^2-b} = c$; then by substituting for y in (2) we obtain

$$a = x^3 + 3x(x^2 - c);$$

that is,
$$4x^3 - 3cx = a.$$

If from this equation the value of x can be determined by trial, the value of y is obtained from $y = x^2 - c$.

Note. We do not here assume $\sqrt{x} + \sqrt{y}$ for the cube root, as in the extraction of the square root; for with this assumption, on cubing we should have

$$a + \sqrt{b} = x\sqrt{x} + 3x\sqrt{y} + 3y\sqrt{x} + y\sqrt{y},$$

and since every term on the right hand side is irrational we cannot equate rational and irrational parts.

Example. Find the cube root of $72 - 32\sqrt{5}$.

Assume $\sqrt[3]{72 - 32\sqrt{5}} = x - \sqrt{y}$;

then $\sqrt[3]{72 + 32\sqrt{5}} = x + \sqrt{y}$.

By multiplication, $\sqrt[3]{5184 - 1024 \times 5} = x^2 - y$;

that is, $4 = x^2 - y$ .... (1).

Again $72 - 32\sqrt{5} = x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y}$;

whence $72 = x^3 + 3xy$ .... (2).

From (1) and (2), $72 = x^3 + 3x(x^2 - 4)$;

that is, $x^3 - 3x = 18$.

By trial, we find that $x = 3$; hence $y = 5$, and the cube root is $3 - \sqrt{5}$.

90. When the binomial whose cube root we are seeking consists of *two* quadratic surds, we proceed as follows.

Example. Find the cube root of $9\sqrt{3} + 11\sqrt{2}$.

$$\sqrt[3]{9\sqrt{3} + 11\sqrt{2}} = \sqrt[3]{3\sqrt{3}\left(3 + \frac{11}{3}\sqrt{\frac{2}{3}}\right)}$$

$$= \sqrt{3}\sqrt[3]{3 + \frac{11}{3}\sqrt{\frac{2}{3}}}.$$

By proceeding as in the last article, we find that

$$\sqrt[3]{3 + \frac{11}{3}\sqrt{\frac{2}{3}}} = 1 + \sqrt{\frac{2}{3}};$$

$\therefore$ the required cube root $= \sqrt{3}\left(1 + \sqrt{\frac{2}{3}}\right)$

$$= \sqrt{3} + \sqrt{2}.$$

91. We add a few harder examples in surds.

Example 1. Express with rational denominator $\dfrac{4}{\sqrt[3]{9} - \sqrt[3]{3} + 1}$.

The expression $= \dfrac{4}{3^{\frac{2}{3}} - 3^{\frac{1}{3}} + 1}$

$$= \frac{4\left(3^{\frac{1}{3}} + 1\right)}{\left(3^{\frac{1}{3}} + 1\right)\left(3^{\frac{2}{3}} - 3^{\frac{1}{3}} + 1\right)}$$

$$= \frac{4\left(3^{\frac{1}{3}} + 1\right)}{3 + 1} = 3^{\frac{1}{3}} + 1.$$

Example 2. Find the square root of

$$\frac{3}{2}(x-1)+\sqrt{2x^2-7x-4}\,.$$

$$\text{The expression} = \frac{1}{2}\left\{3x-3+2\sqrt{(2x+1)(x-4)}\right\}$$

$$= \frac{1}{2}\left\{(2x+1)+(x-4)+2\sqrt{(2x+1)(x-4)}\right\};$$

hence, by inspection, the square root is

$$\frac{1}{\sqrt{2}}\left(\sqrt{2x+1}+\sqrt{x-4}\right).$$

Example 3. Given $\sqrt{5}=2\cdot23607$, find the value of

$$\frac{\sqrt{3-\sqrt{5}}}{\sqrt{2}+\sqrt{7-3\sqrt{5}}}.$$

Multiplying numerator and denominator by $\sqrt{2}$,

$$\text{the expression} = \frac{\sqrt{6-2\sqrt{5}}}{2+\sqrt{14-6\sqrt{5}}}$$

$$= \frac{\sqrt{5}-1}{2+3-\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} = \cdot44721.$$

EXAMPLES. VIII. a.

Express as equivalent fractions with rational denominator:

1. $\dfrac{1}{1+\sqrt{2}-\sqrt{3}}$.

2. $\dfrac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

3. $\dfrac{1}{\sqrt{a}+\sqrt{b}+\sqrt{a+b}}$.

4. $\dfrac{2\sqrt{a+1}}{\sqrt{a-1}-\sqrt{2a}+\sqrt{a+1}}$.

5. $\dfrac{\sqrt{10}+\sqrt{5}-\sqrt{3}}{\sqrt{3}+\sqrt{10}-\sqrt{5}}$.

6. $\dfrac{(\sqrt{3}+\sqrt{5})(\sqrt{5}+\sqrt{2})}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

Find a factor which will rationalise:

7. $\sqrt[3]{3}-\sqrt{2}$.
8. $\sqrt[6]{5}+\sqrt[3]{2}$.
9. $a^{\frac{1}{6}}+b^{\frac{1}{4}}$.
10. $\sqrt[3]{3}-1$.
11. $2+\sqrt[4]{7}$.
12. $\sqrt[3]{5}-\sqrt[4]{3}$.

Express with rational denominator:

13. $\dfrac{\sqrt[3]{3}-1}{\sqrt[3]{3}+1}$. **14.** $\dfrac{\sqrt[6]{9}-\sqrt[6]{8}}{\sqrt[6]{9}+\sqrt[6]{8}}$. **15.** $\dfrac{\sqrt{2}\cdot\sqrt[3]{3}}{\sqrt[3]{3}+\sqrt{2}}$.

16. $\dfrac{\sqrt[3]{3}}{\sqrt{3}+\sqrt[6]{9}}$. **17.** $\dfrac{\sqrt{8}+\sqrt[3]{4}}{\sqrt{8}-\sqrt[3]{4}}$. **18.** $\dfrac{\sqrt[6]{27}}{3-\sqrt[6]{9}}$.

Find the square root of

19. $16-2\sqrt{20}-2\sqrt{28}+2\sqrt{35}$. **20.** $24+4\sqrt{15}-4\sqrt{21}-2\sqrt{35}$.

21. $6+\sqrt{12}-\sqrt{24}-\sqrt{8}$. **22.** $5-\sqrt{10}-\sqrt{15}+\sqrt{6}$.

23. $a+3b+4+4\sqrt{a}-4\sqrt{3b}-2\sqrt{3ab}$.

24. $21+3\sqrt{8}-6\sqrt{3}-6\sqrt{7}-\sqrt{24}-\sqrt{56}+2\sqrt{21}$.

Find the cube root of

25. $10+6\sqrt{3}$. **26.** $38+17\sqrt{5}$. **27.** $99-70\sqrt{2}$.

28. $38\sqrt{14}-100\sqrt{2}$. **29.** $54\sqrt{3}+41\sqrt{5}$. **30.** $135\sqrt{3}-87\sqrt{6}$.

Find the square root of

31. $a+x+\sqrt{2ax+x^2}$. **32.** $2a-\sqrt{3a^2-2ab-b^2}$.

33. $1+a^2+(1+a^2+a^4)^{\frac{1}{2}}$. **34.** $1+(1-a^2)^{-\frac{1}{2}}$.

35. If $a=\dfrac{1}{2-\sqrt{3}}$, $b=\dfrac{1}{2+\sqrt{3}}$, find the value of $7a^2+11ab-7b^2$.

36. If $x=\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$, $y=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, find the value of $3x^2-5xy+3y^2$.

Find the value of

37. $\dfrac{\sqrt{26-15\sqrt{3}}}{5\sqrt{2}-\sqrt{38+5\sqrt{3}}}$. **38.** $\sqrt{\dfrac{6+2\sqrt{3}}{33-19\sqrt{3}}}$.

39. $(28-10\sqrt{3})^{\frac{1}{2}}-(7+4\sqrt{3})^{-\frac{1}{2}}$. **40.** $(26+15\sqrt{3})^{\frac{2}{3}}-(26+15\sqrt{3})^{-\frac{2}{3}}$.

41. Given $\sqrt{5}=2{\cdot}23607$, find the value of

$$\frac{10\sqrt{2}}{\sqrt{18}-\sqrt{3+\sqrt{5}}}-\frac{\sqrt{10}+\sqrt{18}}{\sqrt{8}+\sqrt{3-\sqrt{5}}}.$$

42. Divide $x^3+1+3x\sqrt[3]{2}$ by $x-1+\sqrt[3]{2}$.

43. Find the cube root of $9ab^2+(b^2+24a^2)\sqrt{b^2-3a^2}$.

44. Evaluate $\dfrac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}}$, when $2x=\sqrt{a}+\dfrac{1}{\sqrt{a}}$,

Imaginary Quantities.

92. Although from the rule of signs it is evident that a negative quantity cannot have a real square root, yet imaginary quantities represented by symbols of the form $\sqrt{-a}$, $\sqrt{-1}$ are of frequent occurrence in mathematical investigations, and their use leads to valuable results. We therefore proceed to explain in what sense such roots are to be regarded.

When the quantity under the radical sign is negative, we can no longer consider the symbol $\sqrt{}$ as indicating a possible arithmetical operation; but just as $\sqrt{a}$ may be defined as a symbol which obeys the relation $\sqrt{a} \times \sqrt{a} = a$, so we shall define $\sqrt{-a}$ to be such that $\sqrt{-a} \times \sqrt{-a} = -a$, and we shall accept the meaning to which this assumption leads us.

It will be found that this definition will enable us to bring imaginary quantities under the dominion of ordinary algebraical rules, and that through their use results may be obtained which can be relied on with as much certainty as others which depend solely on the use of real quantities.

93. By definition, $\sqrt{-1} \times \sqrt{-1} = -1.$

$$\therefore \quad \sqrt{a} \,.\, \sqrt{-1} \times \sqrt{a} \,.\, \sqrt{-1} = a(-1);$$

that is, $$(\sqrt{a} \,.\, \sqrt{-1})^2 = -a.$$

Thus the product $\sqrt{a} \,.\, \sqrt{-1}$ may be regarded as equivalent to the imaginary quantity $\sqrt{-a}$.

94. It will generally be found convenient to indicate the imaginary character of an expression by the presence of the symbol $\sqrt{-1}$; thus

$$\sqrt{-4} = \sqrt{4 \times (-1)} = 2\sqrt{-1}.$$

$$\sqrt{-7a^2} = \sqrt{7a^2 \times (-1)} = a\sqrt{7}\sqrt{-1}.$$

95. We shall always consider that, in the absence of any statement to the contrary, of the signs which may be prefixed before a radical the positive sign is to be taken. But in the use of imaginary quantities there is one point of importance which deserves notice.

Since $$(-a)\times(-b)=ab,$$
by taking the square root, we have
$$\sqrt{-a}\times\sqrt{-b}=\pm\sqrt{ab}.$$

Thus in forming the product of $\sqrt{-a}$ and $\sqrt{-b}$ it would appear that either of the signs + or − might be placed before $\sqrt{ab}$. This is not the case, for
$$\begin{aligned}\sqrt{-a}\times\sqrt{-b}&=\sqrt{a}\,.\,\sqrt{-1}\times\sqrt{b}\,.\,\sqrt{-1}\\&=\sqrt{ab}\,(\sqrt{-1})^2\\&=-\sqrt{ab}.\end{aligned}$$

96. It is usual to apply the term 'imaginary' to all expressions which are not wholly real. Thus $a+b\sqrt{-1}$ may be taken as the general type of all imaginary expressions. *Here* a *and* b *are real quantities, but not necessarily rational.*

97. In dealing with imaginary quantities we apply the laws of combination which have been proved in the case of other surd quantities.

Example 1. $a+b\sqrt{-1}\pm(c+d\sqrt{-1})=a\pm c+(b\pm d)\sqrt{-1}.$

Example 2. The product of $a+b\sqrt{-1}$ and $c+d\sqrt{-1}$
$$\begin{aligned}&=(a+b\sqrt{-1})\,(c+d\sqrt{-1})\\&=ac-bd+(bc+ad)\sqrt{-1}.\end{aligned}$$

98. *If* $\mathrm{a}+\mathrm{b}\sqrt{-1}=0$, *then* $\mathrm{a}=0$, *and* $\mathrm{b}=0$.

For, if $$a+b\sqrt{-1}=0,$$
then $$b\sqrt{-1}=-a;$$
$$\therefore\ -b^2=a^2;$$
$$\therefore\ a^2+b^2=0.$$

Now a^2 and b^2 are both positive, therefore their sum cannot be zero unless each of them is separately zero; that is, $a=0$, and $b=0$.

99. *If* $\mathrm{a}+\mathrm{b}\sqrt{-1}=\mathrm{c}+\mathrm{d}\sqrt{-1}$, *then* $\mathrm{a}=\mathrm{c}$, *and* $\mathrm{b}=\mathrm{d}$.

For, by transposition, $a-c+(b-d)\sqrt{-1}=0$;
therefore, by the last article, $a-c=0$, and $b-d=0$;
that is $$a=c,\ \text{and}\ b=d.$$

Thus in order that two imaginary expressions may be equal it is necessary and sufficient that the real parts should be equal, and the imaginary parts should be equal.

100. Definition. When two imaginary expressions differ only in the sign of the imaginary part they are said to be **conjugate.**

Thus $a - b\sqrt{-1}$ is conjugate to $a + b\sqrt{-1}$.

Similarly $\sqrt{2} + 3\sqrt{-1}$ is conjugate to $\sqrt{2} - 3\sqrt{-1}$.

101. *The sum and the product of two conjugate imaginary expressions are both real.*

For $$a + b\sqrt{-1} + a - b\sqrt{-1} = 2a.$$

Again $$(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 - (-b^2)$$
$$= a^2 + b^2.$$

102. Definition. The positive value of the square root of $a^2 + b^2$ is called the **modulus** of each of the conjugate expressions

$$a + b\sqrt{-1} \text{ and } a - b\sqrt{-1}.$$

103. *The modulus of the product of two imaginary expressions is equal to the product of their moduli.*

Let the two expressions be denoted by $a+b\sqrt{-1}$ and $c+d\sqrt{-1}$.

Then their product $= ac - bd + (ad + bc)\sqrt{-1}$, which is an imaginary expression whose modulus

$$= \sqrt{(ac - bd)^2 + (ad + bc)^2}$$
$$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$$
$$= \sqrt{(a^2 + b^2)(c^2 + d^2)}$$
$$= \sqrt{a^2 + b^2} \times \sqrt{c^2 + d^2};$$

which proves the proposition.

104. If the denominator of a fraction is of the form $a + b\sqrt{-1}$, it may be rationalised by multiplying the numerator and the denominator by the conjugate expression $a - b\sqrt{-1}$.

For instance

$$\frac{c+d\sqrt{-1}}{a+b\sqrt{-1}} = \frac{(c+d\sqrt{-1})(a-b\sqrt{-1})}{(a+b\sqrt{-1})(a-b\sqrt{-1})}$$

$$= \frac{ac+bd+(ad-bc)\sqrt{-1}}{a^2+b^2}$$

$$= \frac{ac+bd}{a^2+b^2} + \frac{ad-bc}{a^2+b^2}\sqrt{-1}.$$

Thus by reference to Art. 97, we see that *the sum, difference, product, and quotient of two imaginary expressions is in each case an imaginary expression of the same form.*

105. *To find the square root of* $a + b\sqrt{-1}$.

Assume $\qquad \sqrt{a+b\sqrt{-1}} = x + y\sqrt{-1},$

where x and y are real quantities.

By squaring, $\quad a+b\sqrt{-1} = x^2 - y^2 + 2xy\sqrt{-1}$;

therefore, by equating real and imaginary parts,

$$x^2 - y^2 = a \quad \text{.............................} \quad (1),$$

$$2xy = b \quad \text{.............................} \quad (2);$$

$$\therefore \; (x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2$$

$$= a^2 + b^2;$$

$$\therefore \; x^2 + y^2 = \sqrt{a^2+b^2} \quad \text{.......................} \quad (3).$$

From (1) and (3), we obtain

$$x^2 = \frac{\sqrt{a^2+b^2}+a}{2}, \quad y^2 = \frac{\sqrt{a^2+b^2}-a}{2};$$

$$\therefore \; x = \pm\left\{\frac{\sqrt{a^2+b^2}+a}{2}\right\}^{\frac{1}{2}}, \quad y = \pm\left\{\frac{\sqrt{a^2+b^2}-a}{2}\right\}^{\frac{1}{2}}.$$

Thus the required root is obtained.

Since x and y are real quantities, x^2+y^2 is positive, and therefore in (3) the positive sign must be prefixed before the quantity $\sqrt{a^2+b^2}$.

Also from (2) we see that the product xy must have the same sign as b; hence x and y must have like signs if b is positive, and unlike signs if b is negative.

Example 1. Find the square root of $-7-24\sqrt{-1}$.

Assume $\sqrt{-7-24\sqrt{-1}} = x + y\sqrt{-1}$;

then $-7-24\sqrt{-1} = x^2 - y^2 + 2xy\sqrt{-1}$;

$$\therefore\ x^2 - y^2 = -7 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $2xy = -24$.

$$\therefore\ (x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2$$
$$= 49 + 576$$
$$= 625;$$
$$\therefore\ x^2 + y^2 = 25 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2), $x^2 = 9$ and $y^2 = 16$;

$$\therefore\ x = \pm 3,\ y = \pm 4.$$

Since the product xy is negative, we must take

$$x = 3,\ y = -4;\ \text{ or } x = -3,\ y = 4.$$

Thus the roots are $3 - 4\sqrt{-1}$ and $-3 + 4\sqrt{-1}$;

that is, $\sqrt{-7-24\sqrt{-1}} = \pm(3 - 4\sqrt{-1})$.

Example 2. To find the value of $\sqrt[4]{-64a^4}$.

$$\sqrt[4]{-64a^4} = \sqrt{\pm 8a^2\sqrt{-1}}$$
$$= 2a\sqrt{2}\sqrt{\pm\sqrt{-1}}.$$

It remains to find the value of $\sqrt{\pm\sqrt{-1}}$.

Assume $\sqrt{+\sqrt{-1}} = x + y\sqrt{-1}$;

then $+\sqrt{-1} = x^2 - y^2 + 2xy\sqrt{-1}$;

$$\therefore\ x^2 - y^2 = 0 \text{ and } 2xy = 1;$$

whence $x = \dfrac{1}{\sqrt{2}},\ y = \dfrac{1}{\sqrt{2}};$ or $x = -\dfrac{1}{\sqrt{2}},\ y = -\dfrac{1}{\sqrt{2}};$

$$\therefore\ \sqrt{+\sqrt{-1}} = \pm\frac{1}{\sqrt{2}}(1+\sqrt{-1}).$$

Similarly $\sqrt{-\sqrt{-1}} = \pm\dfrac{1}{\sqrt{2}}(1-\sqrt{-1})$

$$\therefore\ \sqrt{\pm\sqrt{-1}} = \pm\frac{1}{\sqrt{2}}(1\pm\sqrt{-1});$$

and finally $\sqrt[4]{-64a^4} = \pm 2a(1 \pm \sqrt{-1})$.

106. The symbol $\sqrt{-1}$ is often represented by the letter i; but until the student has had a little practice in the use of imaginary quantities he will find it easier to retain the symbol $\sqrt{-1}$. It is useful to notice the successive powers of $\sqrt{-1}$ or i; thus

$$(\sqrt{-1})^1 = \sqrt{-1}, \qquad i = i;$$
$$(\sqrt{-1})^2 = -1, \qquad i^2 = -1;$$
$$(\sqrt{-1})^3 = -\sqrt{-1}, \qquad i^3 = -i;$$
$$(\sqrt{-1})^4 = 1, \qquad i^4 = 1;$$

and since each power is obtained by multiplying the one before it by $\sqrt{-1}$, or i, we see that the results must now recur.

107. We shall now investigate the properties of certain imaginary quantities which are of very frequent occurrence.

Suppose $\qquad x = \sqrt[3]{1}$; then $x^3 = 1$, or $x^3 - 1 = 0$;

that is, $\qquad (x-1)(x^2+x+1) = 0.$

$\therefore$ either $\qquad x - 1 = 0$, or $x^2 + x + 1 = 0$;

whence $\qquad x = 1$, or $\quad x = \dfrac{-1 \pm \sqrt{-3}}{2}$.

It may be shewn by actual involution that each of these values when cubed is equal to unity. Thus unity has three cube roots,

$$1, \quad \frac{-1+\sqrt{-3}}{2}, \quad \frac{-1-\sqrt{-3}}{2};$$

two of which are imaginary expressions.

Let us denote these by α and β; then since they are the roots of the equation

$$x^2 + x + 1 = 0,$$

their product is equal to unity;

that is, $\qquad \alpha\beta = 1;$

$$\therefore \alpha^3\beta = \alpha^2;$$

that is, $\qquad \beta = \alpha^2$, since $\alpha^3 = 1$.

Similarly we may shew that $\alpha = \beta^2$.

108. Since *each of the imaginary roots is the square of the other*, it is usual to denote the three cube roots of unity by $1, \omega, \omega^2$.

Also ω satisfies the equation $x^2+x+1=0$;

$$\therefore\ 1+\omega+\omega^2=0;$$

that is, *the sum of the three cube roots of unity is zero.*

Again, $\omega\,.\,\omega^2=\omega^3=1$;

therefore (1) *the product of the two imaginary roots is unity;*
(2) *every integral power of ω^3 is unity.*

109. It is useful to notice that the successive positive integral powers of ω are 1, ω, and ω^2; for, if n be a multiple of 3, it must be of the form $3m$; and $\omega^n=\omega^{3m}=1$.

If n be not a multiple of 3, it must be of the form $3m+1$ or $3m+2$.

If $n=3m+1$, $\omega^n=\omega^{3m+1}=\omega^{3m}\,.\,\omega=\omega$.

If $n=3m+2$, $\omega^n=\omega^{3m+2}=\omega^{3m}\,.\,\omega^2=\omega^2$.

110. We now see that every quantity has three cube roots, two of which are imaginary. For the cube roots of a^3 are those of $a^3\times 1$, and therefore are a, $a\omega$, $a\omega^2$. Similarly the cube roots of 9 are $\sqrt[3]{9}$, $\omega\sqrt[3]{9}$, $\omega^2\sqrt[3]{9}$, where $\sqrt[3]{9}$ is the cube root found by the ordinary arithmetical rule. In future, unless otherwise stated, the symbol $\sqrt[3]{a}$ will always be taken to denote the arithmetical cube root of a.

Example 1. Reduce $\dfrac{(2+3\sqrt{-1})^2}{2+\sqrt{-1}}$ to the form $A+B\sqrt{-1}$.

The expression

$$\begin{aligned}&=\frac{4-9+12\sqrt{-1}}{2+\sqrt{-1}}\\&=\frac{(-5+12\sqrt{-1})(2-\sqrt{-1})}{(2+\sqrt{-1})(2-\sqrt{-1})}\\&=\frac{-10+12+29\sqrt{-1}}{4+1}\\&=\frac{2}{5}+\frac{29}{5}\sqrt{-1};\end{aligned}$$

which is of the required form.

Example 2. *Resolve* x^3+y^3 *into three factors of the first degree.*

Since $x^3+y^3=(x+y)(x^2-xy+y^2)$

$$\therefore\ x^3+y^3=(x+y)(x+\omega y)(x+\omega^2 y);$$

for $\omega+\omega^2=-1$, and ω^3-1.

Example 3. Shew that

$$(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)=a^2+b^2+c^2-bc-ca-ab.$$

In the product of $\quad a+\omega b+\omega^2 c$ and $a+\omega^2 b+\omega c$,

the coefficients of b^2 and c^2 are ω^3, or 1;

the coefficient of $bc \qquad =\omega^2+\omega^4=\omega^2+\omega=-1$;

the coefficients of ca and $ab=\omega^2+\omega=-1$;

$$\therefore\ (a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)=a^2+b^2+c^2-bc-ca-ab.$$

Example 4. Shew that

$$(1+\omega-\omega^2)^3-(1-\omega+\omega^2)^3=0.$$

Since $\qquad 1+\omega+\omega^2=0$, we have

$$\begin{aligned}(1+\omega-\omega^2)^3-(1-\omega+\omega^2)^3&=(-2\omega^2)^3-(-2\omega)^3\\&=-8\omega^6+8\omega^3\\&=-8+8\\&=0.\end{aligned}$$

EXAMPLES. VIII. b.

1. Multiply $2\sqrt{-3}+3\sqrt{-2}$ by $4\sqrt{-3}-5\sqrt{-2}$.

2. Multiply $3\sqrt{-7}-5\sqrt{-2}$ by $3\sqrt{-7}+5\sqrt{-2}$.

3. Multiply $e^{\sqrt{-1}}+e^{-\sqrt{-1}}$ by $e^{\sqrt{-1}}-e^{-\sqrt{-1}}$.

4. Multiply $x-\dfrac{1+\sqrt{-3}}{2}$ by $x-\dfrac{1-\sqrt{-3}}{2}$.

Express with rational denominator:

5. $\dfrac{1}{3-\sqrt{-2}}$. $\qquad$ 6. $\dfrac{3\sqrt{-2}+2\sqrt{-5}}{3\sqrt{-2}-2\sqrt{-5}}$.

7. $\dfrac{3+2\sqrt{-1}}{2-5\sqrt{-1}}+\dfrac{3-2\sqrt{-1}}{2+5\sqrt{-1}}$. $\qquad$ 8. $\dfrac{a+x\sqrt{-1}}{a-x\sqrt{-1}}-\dfrac{a-x\sqrt{-1}}{a+x\sqrt{-1}}$.

9. $\dfrac{(x+\sqrt{-1})^2}{x-\sqrt{-1}}-\dfrac{(x-\sqrt{-1})^2}{x+\sqrt{-1}}$. $\qquad$ 10. $\dfrac{(a+\sqrt{-1})^3-(a-\sqrt{-1})^3}{(a+\sqrt{-1})^2-(a-\sqrt{-1})^2}$.

11. Find the value of $(-\sqrt{-1})^{4n+3}$, when n is a positive integer.

12. Find the square of $\sqrt{9+40\sqrt{-1}}+\sqrt{9-40\sqrt{-1}}$.

Find the square root of

13. $-5+12\sqrt{-1}$.
14. $-11-60\sqrt{-1}$.
15. $-47+8\sqrt{-3}$.
16. $-8\sqrt{-1}$.
17. $a^2-1+2a\sqrt{-1}$.
18. $4ab-2(a^2-b^2)\sqrt{-1}$.

Express in the form $A+iB$

19. $\dfrac{3+5i}{2-3i}$.
20. $\dfrac{\sqrt{3}-i\sqrt{2}}{2\sqrt{3}-i\sqrt{2}}$.
21. $\dfrac{1+i}{1-i}$.
22. $\dfrac{(1+i)^2}{3-i}$.
23. $\dfrac{(a+ib)^2}{a-ib}-\dfrac{(a-ib)^2}{a+ib}$.

If 1, ω, ω^2 are the three cube roots of unity, prove

24. $(1+\omega^2)^4=\omega$.
25. $(1-\omega+\omega^2)(1+\omega-\omega^2)=4$.
26. $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)=9$.
27. $(2+5\omega+2\omega^2)^6=(2+2\omega+5\omega^2)^6=729$.
28. $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)\ldots$ to $2n$ factors $=2^{2n}$.
29. Prove that

$$x^3+y^3+z^3-3xyz=(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega).$$

30. If $\quad x=a+b,\ \ y=a\omega+b\omega^2,\ \ z=a\omega^2+b\omega,$

shew that

(1) $xyz=a^3+b^3$.

(2) $x^2+y^2+z^2=6ab$.

(3) $x^3+y^3+z^3=3(a^3+b^3)$.

31. If $\ ax+cy+bz=X,\ cx+by+az=Y,\ bx+ay+cz=Z,$
shew that $(a^2+b^2+c^2-bc-ca-ab)(x^2+y^2+z^2-yz-zx-xy)$
$$=X^2+Y^2+Z^2-YZ-XZ-XY.$$

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

CHAPTER IX.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

THE THEORY OF QUADRATIC EQUATIONS.

111. After suitable reduction every quadratic equation may be written in the form

$$ax^2 + bx + c = 0 \quad \text{.....................} \quad (1),$$

and the solution of the equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{.......................} \quad (2).$$

We shall now prove some important propositions connected with the roots and coefficients of all equations of which (1) is the type.

112. *A quadratic equation cannot have more than two roots.*

For, if possible, let the equation $ax^2 + bx + c = 0$ have three *different* roots α, β, γ. Then since each of these values must satisfy the equation, we have

$$a\alpha^2 + b\alpha + c = 0 \quad \text{.......................} \quad (1),$$

$$a\beta^2 + b\beta + c = 0 \quad \text{.......................} \quad (2),$$

$$a\gamma^2 + b\gamma + c = 0 \quad \text{.......................} \quad (3).$$

From (1) and (2), by subtraction,

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0;$$

divide out by $\alpha - \beta$ which, by hypothesis, is not zero; then

$$a(\alpha + \beta) + b = 0.$$

Similarly from (2) and (3)

$$a(\beta + \gamma) + b = 0;$$

$\therefore$ by subtraction $\qquad a(\alpha - \gamma) = 0;$

which is impossible, since, by hypothesis, a is not zero, and α is not equal to γ. Hence there cannot be three different roots.

113. In Art. 111 let the two roots in (2) be denoted by α and β, so that

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a};$$

then we have the following results:

(1) If $b^2 - 4ac$ (the quantity under the radical) is positive, α and β are real and unequal.

(2) If $b^2 - 4ac$ is zero, α and β are real and equal, each reducing in this case to $-\frac{b}{2a}$.

(3) If $b^2 - 4ac$ is negative, α and β are imaginary and unequal.

(4) If $b^2 - 4ac$ is a perfect square, α and β are rational and unequal.

By applying these tests the nature of the roots of any quadratic may be determined without solving the equation.

Example 1. Shew that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real values of x.

Here $a = 2$, $b = -6$, $c = 7$; so that

$$b^2 - 4ac = (-6)^2 - 4 \cdot 2 \cdot 7 = -20.$$

Therefore the roots are imaginary.

Example 2. If the equation $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find k.

The condition for equal roots gives

$$(k+2)^2 = 9k,$$
$$k^2 - 5k + 4 = 0,$$
$$(k-4)(k-1) = 0;$$
$$\therefore\ k = 4, \text{ or } 1.$$

Example 3. Shew that the roots of the equation

$$x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$$

are rational.

The roots will be rational provided $(-2p)^2 - 4(p^2 - q^2 + 2qr - r^2)$ is a perfect square. But this expression reduces to $4(q^2 - 2qr + r^2)$, or $4(q-r)^2$. Hence the roots are rational.

114. Since $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$,

we have by addition

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= -\frac{2b}{2a} = -\frac{b}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and by multiplication we have

$$\alpha\beta = \frac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{4a^2}$$

$$= \frac{(-b)^2-(b^2-4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a} \quad \text{.......................................} \quad (2).$$

By writing the equation in the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

these results may also be expressed as follows.

In a quadratic equation *where the coefficient of the first term is unity,*

(i) the sum of the roots is equal to the coefficient of x with its sign changed;

(ii) the product of the roots is equal to the third term.

NOTE. In any equation the term which does not contain the unknown quantity is frequently called *the absolute term.*

115. Since $\quad -\frac{b}{a} = \alpha + \beta, \text{ and } \frac{c}{a} = \alpha\beta,$

the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ may be written

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad \text{......................} (1).$$

Hence any quadratic may also be expressed in the form

$$x^2 - (\text{sum of roots})\, x + \text{product of roots} = 0 \quad \text{.........} (2).$$

Again, from (1) we have

$$(x-\alpha)(x-\beta) = 0 \quad \text{........................} (3).$$

We may now easily form an equation with given roots.

Example 1. Form the equation whose roots are 3 and -2.

The equation is $\quad (x-3)(x+2)=0,$

or $\quad x^2 - x - 6 = 0.$

When the roots are irrational it is easier to use the following method.

Example 2. Form the equation whose roots are $2+\sqrt{3}$ and $2-\sqrt{3}$.

We have
$$\text{sum of roots}=4,$$
$$\text{product of roots}=1;$$
$\therefore$ the equation is
$$x^2-4x+1=0,$$
by using formula (2) of the present article.

116. By a method analogous to that used in Example 1 of the last article we can form an equation with three or more given roots.

Example 1. Form the equation whose roots are 2, -3, and $\frac{7}{5}$.

The required equation must be satisfied by each of the following suppositions:
$$x-2=0,\ \ x+3=0,\ \ x-\frac{7}{5}=0;$$
therefore the equation must be
$$(x-2)(x+3)\left(x-\frac{7}{5}\right)=0;$$
that is,
$$(x-2)(x+3)(5x-7)=0,$$
or
$$5x^3-2x^2-37x+42=0.$$

Example 2. Form the equation whose roots are 0, $\pm a$, $\frac{c}{b}$.

The equation has to be satisfied by
$$x=0,\ \ x=a,\ \ x=-a,\ \ x=\frac{c}{b};$$
therefore it is
$$x(x+a)(x-a)\left(x-\frac{c}{b}\right)=0;$$
that is,
$$x(x^2-a^2)(bx-c)=0,$$
or
$$bx^4-cx^3-a^2bx^2+a^2cx=0.$$

117. The results of Art. 114 are most important, and they are generally sufficient to solve problems connected with the roots of quadratics. In such questions *the roots should never be considered singly*, but use should be made of the relations obtained by writing down the sum of the roots, and their product, in terms of the coefficients of the equation.

Example 1. If α and β are the roots of $x^2-px+q=0$, find the value of (1) $\alpha^2+\beta^2$, (2) $\alpha^3+\beta^3$.

We have
$$\alpha+\beta=p,$$
$$\alpha\beta=q.$$
$$\therefore\ \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$=p^2-2q.$$

Again,
$$\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)$$
$$=p\{(\alpha+\beta)^2-3\alpha\beta\}$$
$$=p(p^2-3q).$$

Example 2. If α, β are the roots of the equation $lx^2+mx+n=0$, find the equation whose roots are $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$.

We have
$$\text{sum of roots}=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta},$$
$$\text{product of roots}=\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=1;$$

$\therefore$ by Art. 115 the required equation is
$$x^2-\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)x+1=0,$$
or
$$\alpha\beta x^2-(\alpha^2+\beta^2)x+\alpha\beta=0.$$

As in the last example $\alpha^2+\beta^2=\frac{m^2-2nl}{l^2}$, and $\alpha\beta=\frac{n}{l}$.

$\therefore$ the equation is
$$\frac{n}{l}x^2-\frac{m^2-2nl}{l^2}x+\frac{n}{l}=0,$$
or
$$nlx^2-(m^2-2nl)x+nl=0.$$

Example 3. When $x=\frac{3+5\sqrt{-1}}{2}$, find the value of $2x^3+2x^2-7x+72$; and shew that it will be unaltered if $\frac{3-5\sqrt{-1}}{2}$ be substituted for x.

Form the quadratic equation whose roots are $\frac{3\pm5\sqrt{-1}}{2}$;

the sum of the roots $=3$;

the product of the roots $=\frac{17}{2}$;

hence the equation is $\qquad 2x^2-6x+17=0$;

$\therefore$ $2x^2-6x+17$ is a quadratic *expression* which vanishes for *either* of the values $\frac{3\pm5\sqrt{-1}}{2}$.

Now
$$2x^3+2x^2-7x+72=x(2x^2-6x+17)+4(2x^2-6x+17)+4$$
$$=x\times0+4\times0+4$$
$$=4;$$

which is the numerical value of the expression in each of the supposed cases.

118. *To find the condition that the roots of the equation* $ax^2 + bx + c = 0$ *should be* (1) *equal in magnitude and opposite in sign,* (2) *reciprocals.*

The roots will be equal in magnitude and opposite in sign if their sum is zero; hence the required condition is

$$-\frac{b}{a} = 0, \text{ or } b = 0.$$

Again, the roots will be reciprocals when their product is unity; hence we must have

$$\frac{c}{a} = 1, \text{ or } c = a.$$

The first of these results is of frequent occurrence in Analytical Geometry, and the second is a particular case of a more general condition applicable to equations of any degree.

Example. Find the condition that the roots of $ax^2 + bx + c = 0$ may be (1) both positive, (2) opposite in sign, but the greater of them negative.

We have $$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

(1) If the roots are both positive, $\alpha\beta$ is positive, and therefore c and a have like signs.

Also, since $\alpha + \beta$ is positive, $\frac{b}{a}$ is negative; therefore b and a have unlike signs.

Hence the required condition is that the signs of a and c should be like, and opposite to the sign of b.

(2) If the roots are of opposite signs, $\alpha\beta$ is negative, and therefore c and a have unlike signs.

Also since $\alpha + \beta$ has the sign of the greater root it is negative, and therefore $\frac{b}{a}$ is positive; therefore b and a have like signs.

Hence the required condition is that the signs of a and b should be like, and opposite to the sign of c.

EXAMPLES. IX. a.

Form the equations whose roots are

1. $-\frac{4}{5}, \frac{3}{7}$. 2. $\frac{m}{n}, -\frac{n}{m}$. 3. $\frac{p-q}{p+q}, -\frac{p+q}{p-q}$.

4. $7 \pm 2\sqrt{5}$. 5. $\pm 2\sqrt{3} - 5$. 6. $-p \pm 2\sqrt{2q}$.

7. $-3 \pm 5i$. **8.** $-a \pm ib$. **9.** $\pm i(a-b)$.

10. $-3, \frac{2}{3}, \frac{1}{2}$. **11.** $\frac{a}{2}, 0, -\frac{2}{a}$. **12.** $2 \pm \sqrt{3}, 4$.

13. Prove that the roots of the following equations are real:

(1) $x^2 - 2ax + a^2 - b^2 - c^2 = 0$,

(2) $(a-b+c)x^2 + 4(a-b)x + (a-b-c) = 0$.

14. If the equation $x^2 - 15 - m(2x - 8) = 0$ has equal roots, find the values of m.

15. For what values of m will the equation

$$x^2 - 2x(1+3m) + 7(3+2m) = 0$$

have equal roots?

16. For what value of m will the equation

$$\frac{x^2 - bx}{ax - c} = \frac{m-1}{m+1}$$

have roots equal in magnitude but opposite in sign?

17. Prove that the roots of the following equations are rational:

(1) $(a+c-b)x^2 + 2cx + (b+c-a) = 0$,

(2) $abc^2x^2 + 3a^2cx + b^2cx - 6a^2 - ab + 2b^2 = 0$.

If α, β are the roots of the equation $ax^2 + bx + c = 0$, find the values of

18. $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$. **19.** $\alpha^4\beta^7 + \alpha^7\beta^4$. **20.** $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$.

Find the value of

$i = \sqrt{-1}$

21. $x^3 + x^2 - x + 22$ when $x = 1 + 2i$.

22. $x^3 - 3x^2 - 8x + 15$ when $x = 3 + i$.

23. $x^3 - ax^2 + 2a^2x + 4a^3$ when $\frac{x}{a} = 1 - \sqrt{-3}$.

24. If α and β are the roots of $x^2 + px + q = 0$, form the equation whose roots are $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$.

25. Prove that the roots of $(x-a)(x-b) = h^2$ are always real.

26. If x_1, x_2 are the roots of $ax^2 + bx + c = 0$, find the value of

(1) $(ax_1 + b)^{-2} + (ax_2 + b)^{-2}$,

(2) $(ax_1 + b)^{-3} + (ax_2 + b)^{-3}$.

27. Find the condition that one root of $ax^2+bx+c=0$ shall be n times the other.

28. If a, β are the roots of $ax^2+bx+c=0$, form the equation whose roots are $a^2+\beta^2$ and $a^{-2}+\beta^{-2}$.

29. Form the equation whose roots are the squares of the sum and of the difference of the roots of

$$2x^2+2(m+n)x+m^2+n^2=0.$$

30. Discuss the signs of the roots of the equation

$$px^2+qx+r=0.$$

119. The following example illustrates a useful application of the results proved in Art. 113.

Example. If x is a real quantity, prove that the expression $\frac{x^2+2x-11}{2(x-3)}$ can have all numerical values except such as lie between 2 and 6.

Let the given expression be represented by y, so that

$$\frac{x^2+2x-11}{2(x-3)}=y;$$

then multiplying up and transposing, we have

$$x^2+2x(1-y)+6y-11=0.$$

This is a quadratic equation, and in order that x may have real values $4(1-y)^2-4(6y-11)$ must be positive; or dividing by 4 and simplifying, $y^2-8y+12$ must be positive; that is, $(y-6)(y-2)$ must be positive. Hence the factors of this product must be both positive, or both negative. In the former case y is greater than 6; in the latter y is less than 2. Therefore y cannot lie between 2 and 6, but may have any other value.

In this example it will be noticed that the *quadratic expression* $y^2-8y+12$ is positive so long as y does not lie between the roots of the corresponding *quadratic equation* $y^2-8y+12=0$.

This is a particular case of the general proposition investigated in the next article.

120. *For all real values of* x *the expression* ax^2+bx+c *has the same sign as* a, *except when the roots of the equation* $ax^2+bx+c=0$ *are real and unequal, and* x *has a value lying between them.*

Case I. Suppose that the roots of the equation

$$ax^2+bx+c=0$$

are real; denote them by α and β, and let α be the greater.

Then $\quad ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

$$= a\{x^2 - (\alpha + \beta)x + \alpha\beta\}$$

$$= a(x - \alpha)(x - \beta).$$

Now if x is greater than α, the factors $x - \alpha$, $x - \beta$ are both positive; and if x is less than β, the factors $x - \alpha$, $x - \beta$ are both negative; therefore in each case the expression $(x - \alpha)(x - \beta)$ is positive, and $ax^2 + bx + c$ has the same sign as a. But if x has a value lying between α and β, the expression $(x - \alpha)(x - \beta)$ is negative, and the sign of $ax^2 + bx + c$ is opposite to that of a.

Case II. If α and β are equal, then

$$ax^2 + bx + c = a(x - \alpha)^2,$$

and $(x - \alpha)^2$ is positive for all real values of x; hence $ax^2 + bx + c$ has the same sign as a.

Case III. Suppose that the equation $ax^2 + bx + c = 0$ has imaginary roots; then

$$ax^2 + bx + c = a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}.$$

But $b^2 - 4ac$ is negative since the roots are imaginary; hence $\frac{4ac - b^2}{4a^2}$ is positive, and the expression

$$\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$$

is positive for all real values of x; therefore $ax^2 + bx + c$ has the same sign as a. This establishes the proposition.

121. From the preceding article it follows that the expression $ax^2 + bx + c$ will always have the same sign whatever real value x may have, provided that $b^2 - 4ac$ is negative or zero; and if this condition is satisfied the expression is positive or negative according as a is positive or negative.

Conversely, in order that the expression $ax^2 + bx + c$ may be always positive, $b^2 - 4ac$ must be negative or zero, and a must be positive; and in order that $ax^2 + bx + c$ may be always negative $b^2 - 4ac$ must be negative or zero, and a must be negative.

Example. Find the limits between which a must lie in order that

$$\frac{ax^2-7x+5}{5x^2-7x+a}$$

may be capable of all values, x being any real quantity.

Put $$\frac{ax^2-7x+5}{5x^2-7x+a}=y;$$

then $$(a-5y)x^2-7x(1-y)+(5-ay)=0.$$

In order that the values of x found from this quadratic may be real, the expression

$$49(1-y)^2-4(a-5y)(5-ay) \text{ must be positive,}$$

that is, $(49-20a)y^2+2(2a^2+1)y+(49-20a)$ must be positive;

hence $(2a^2+1)^2-(49-20a)^2$ must be negative or zero, and $49-20a$ must be positive.

Now $(2a^2+1)^2-(49-20a)^2$ is negative or zero, according as

$$2(a^2-10a+25)\times 2(a^2+10a-24) \text{ is negative or zero;}$$

that is, according as $4(a-5)^2(a+12)(a-2)$ is negative or zero.

This expression is negative as long as a lies between 2 and -12, and for such values $49-20a$ is positive; the expression is zero when $a=5$, -12, or 2, but $49-20a$ is negative when $a=5$. Hence the limiting values are 2 and -12, and a may have any intermediate value.

EXAMPLES. IX. b.

1. Determine the limits between which n must lie in order that the equation

$$2ax(ax+nc)+(n^2-2)c^2=0$$

may have real roots.

2. If x be real, prove that $\dfrac{x}{x^2-5x+9}$ must lie between 1 and $-\dfrac{1}{11}$.

3. Shew that $\dfrac{x^2-x+1}{x^2+x+1}$ lies between 3 and $\dfrac{1}{3}$ for all real values of x.

4. If x be real, prove that $\dfrac{x^2+34x-71}{x^2+2x-7}$ can have no value between 5 and 9.

5. Find the equation whose roots are $\dfrac{\sqrt{a}}{\sqrt{a}\pm\sqrt{a-b}}$.

6. If α, β are roots of the equation $x^2-px+q=0$, find the value of

(1) $\alpha^2(\alpha^2\beta^{-1}-\beta)+\beta^2(\beta^2\alpha^{-1}-\alpha)$,

(2) $(\alpha-p)^{-4}+(\beta-p)^{-4}$.

7. If the roots of $lx^2+nx+n=0$ be in the ratio of $p:q$, prove that

$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0.$$

8. If x be real, the expression $\frac{(x+m)^2-4mn}{2(x-n)}$ admits of all values except such as lie between $2n$ and $2m$.

9. If the roots of the equation $ax^2+2bx+c=0$ be α and β, and those of the equation $Ax^2+2Bx+C=0$ be $\alpha+\delta$ and $\beta+\delta$, prove that

$$\frac{b^2-ac}{a^2}=\frac{B^2-AC}{A^2}.$$

10. Shew that the expression $\frac{px^2+3x-4}{p+3x-4x^2}$ will be capable of all values when x is real, provided that p has any value between 1 and 7.

11. Find the greatest value of $\frac{x+2}{2x^2+3x+6}$ for real values of x.

12. Shew that if x is real, the expression

$$(x^2-bc)(2x-b-c)^{-1}$$

has no real values between b and c.

13. If the roots of $ax^2+2bx+c=0$ be possible and different, then the roots of

$$(a+c)(ax^2+2bx+c)=2(ac-b^2)(x^2+1)$$

will be impossible, and *vice versâ*.

14. Shew that the expression $\frac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$ will be capable of all values when x is real, if a^2-b^2 and c^2-d^2 have the same sign.

*122. We shall conclude this chapter with some miscellaneous theorems and examples. It will be convenient here to introduce a phraseology and notation which the student will frequently meet with in his mathematical reading.

Definition. Any expression which involves x, and whose value is dependent on that of x, is called a **function of x.** Functions of x are usually denoted by symbols of the form $f(x)$, $F(x)$, $\phi(x)$.

Thus the equation $y=f(x)$ may be considered as equivalent to a statement that any change made in the value of x will produce a consequent change in y, and *vice versâ*. The quantities x and y are called **variables**, and are further distinguished as the **independent variable** and the **dependent variable**.

$x^2 + bx + c$

An *independent variable* is a quantity which may have any value we choose to assign to it, and the corresponding *dependent variable* has its value determined as soon as the value of the independent variable is known.

*123. An expression of the form

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$$

where n is a positive integer, and the coefficients $p_0, p_1, p_2, \ldots p_n$ do not involve x, is called a **rational and integral algebraical function of x.** In the present chapter we shall confine our attention to functions of this kind.

*124. A function is said to be **linear** when it contains no higher power of the variable than the first; thus $ax + b$ is a linear function of x. A function is said to be **quadratic** when it contains no higher power of the variable than the second; thus $ax^2 + bx + c$ is a quadratic function of x. Functions of the *third*, *fourth*,... degrees are those in which the highest power of the variable is respectively the *third*, *fourth*,.... Thus in the last article the expression is a function of x of the n^{th} degree.

*125. The symbol $f(x, y)$ is used to denote a function of two variables x and y; thus $ax + by + c$, and $ax^2 + bxy + cy^2 + dx + ey + f$ are respectively linear and quadratic functions of x, y.

The *equations* $f(x) = 0$, $f(x, y) = 0$ are said to be linear, quadratic,... according as the *functions* $f(x)$, $f(x, y)$ are linear, quadratic,....

*126. We have proved in Art. 120 that the *expression* $ax^2 + bx + c$ admits of being put in the form $a(x - \alpha)(x - \beta)$, where α and β are the roots of the *equation* $ax^2 + bx + c = 0$.

Thus a quadratic expression $ax^2 + bx + c$ is capable of being resolved into two rational factors of the first degree, whenever the equation $ax^2 + bx + c = 0$ has rational roots; that is, when $b^2 - 4ac$ is a perfect square.

*127. *To find the condition that a quadratic function of* x, y *may be resolved into two linear factors.*

Denote the function by $f(x, y)$ where

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$$

Write this in descending powers of x, and equate it to zero; thus

$$ax^2 + 2x(hy + g) + by^2 + 2fy + c = 0.$$

Solving this quadratic in x we have

$$x = \frac{-(hy+g) \pm \sqrt{(hy+g)^2 - a(by^2 + 2fy + c)}}{a},$$

or $$ax + hy + g = \pm\sqrt{y^2(h^2 - ab) + 2y(hg - af) + (g^2 - ac)}.$$

Now in order that $f(x, y)$ may be the product of two linear factors of the form $px + qy + r$, the quantity under the radical must be a perfect square; hence

$$(hg - af)^2 = (h^2 - ab)(g^2 - ac).$$

Transposing and dividing by a, we obtain

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0;$$

which is the condition required.

This proposition is of great importance in Analytical Geometry.

*128. To find the condition that the equations

$$ax^2 + bx + c = 0, \quad a'x^2 + b'x + c' = 0$$

may have a common root.

Suppose these equations are both satisfied by $x = \alpha$; then

$$a\alpha^2 + b\alpha + c = 0,$$

$$a'\alpha^2 + b'\alpha + c' = 0;$$

$\therefore$ by cross multiplication

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{ca' - c'a} = \frac{1}{ab' - a'b}.$$

To eliminate α, square the second of these equal ratios and equate it to the product of the other two; thus

$$\frac{\alpha^2}{(ca' - c'a)^2} = \frac{\alpha^2}{(bc' - b'c)} \cdot \frac{1}{(ab' - a'b)};$$

$$\therefore \; (ca' - c'a)^2 = (bc' - b'c)(ab' - a'b),$$

which is the condition required.

It is easy to prove that this is the condition that the two quadratic functions $ax^2 + bxy + cy^2$ and $a'x^2 + b'xy + c'y^2$ may have a common linear factor.

*EXAMPLES. IX. c.

1. For what values of m will the expression

$$y^2 + 2xy + 2x + my - 3$$

be capable of resolution into two rational factors?

2. Find the values of m which will make $2x^2 + mxy + 3y^2 - 5y - 2$ equivalent to the product of two linear factors.

3. Shew that the expression

$$A(x^2 - y^2) - xy(B - C)$$

always admits of two real linear factors.

4. If the equations

$$x^2 + px + q = 0, \quad x^2 + p'x + q' = 0$$

have a common root, shew that it must be either

$$\frac{pq' - p'q}{q - q'} \text{ or } \frac{q - q'}{p' - p}.$$

5. Find the condition that the expressions

$$lx^2 + mxy + ny^2, \quad l'x^2 + m'xy + n'y^2$$

may have a common linear factor.

6. If the expression

$$3x^2 + 2Pxy + 2y^2 + 2ax - 4y + 1$$

can be resolved into linear factors, prove that P must be one of the roots of the equation $P^2 + 4aP + 2a^2 + 6 = 0$.

7. Find the condition that the expressions

$$ax^2 + 2hxy + by^2, \quad a'x^2 + 2h'xy + b'y^2$$

may be respectively divisible by factors of the form $y - mx$, $my + x$.

8. Shew that in the equation

$$x^2 - 3xy + 2y^2 - 2x - 3y - 35 = 0,$$

for every real value of x there is a real value of y, and for every real value of y there is a real value of x.

9. If x and y are two real quantities connected by the equation

$$9x^2 + 2xy + y^2 - 92x - 20y + 244 = 0,$$

then will x lie between 3 and 6, and y between 1 and 10.

10. If $(ax^2 + bx + c)y + a'x^2 + b'x + c' = 0$, find the condition that x may be a rational function of y.

CHAPTER X.

MISCELLANEOUS EQUATIONS.

129. In this chapter we propose to consider some miscellaneous equations; it will be seen that many of these can be solved by the ordinary rules for quadratic equations, but others require some special artifice for their solution.

Example 1. Solve $8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}} = 63.$

Multiply by $x^{\frac{3}{2n}}$ and transpose; thus

$$8x^{\frac{3}{n}} - 63x^{\frac{3}{2n}} - 8 = 0;$$

$$(x^{\frac{3}{2n}} - 8)(8x^{\frac{3}{2n}} + 1) = 0;$$

$$x^{\frac{3}{2n}} = 8, \text{ or } -\frac{1}{8};$$

$$x = (2^3)^{\frac{2n}{3}}, \text{ or } \left(-\frac{1}{2^3}\right)^{\frac{2n}{3}};$$

$$\therefore\ x = 2^{2n}, \text{ or } \frac{1}{2^{2n}}.$$

Example 2. Solve $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}.$

Let $\sqrt{\frac{x}{a}} = y$; then $\sqrt{\frac{a}{x}} = \frac{1}{y}$;

$$\therefore\ 2y + \frac{3}{y} = \frac{b}{a} + \frac{6a}{b};$$

$$2aby^2 - 6a^2y - b^2y + 3ab = 0;$$

$$(2ay - b)(by - 3a) = 0;$$

$$y = \frac{b}{2a}, \text{ or } \frac{3a}{b};$$

$$\therefore\ \frac{x}{a} = \frac{b^2}{4a^2}, \text{ or } \frac{9a^2}{b^2};$$

that is, $x = \frac{b^2}{4a}, \text{ or } \frac{9a^3}{b^2}.$

Example 3. Solve $(x-5)(x-7)(x+6)(x+4)=504$.

We have $$(x^2-x-20)(x^2-x-42)=504;$$

which, being arranged as a quadratic in x^2-x, gives

$$(x^2-x)^2-62(x^2-x)+336=0;$$

$$\therefore\ (x^2-x-6)(x^2-x-56)=0;$$

$$\therefore\ x^2-x-6=0,\ \text{or}\ x^2-x-56=0;$$

whence $$x=3,\ -2,\ 8,\ -7.$$

130. Any equation which can be thrown into the form

$$ax^2+bx+c+p\sqrt{ax^2+bx+c}=q$$

may be solved as follows. Putting $y=\sqrt{ax^2+bx+c}$, we obtain

$$y^2+py-q=0.$$

Let α and β be the roots of this equation, so that

$$\sqrt{ax^2+bx+c}=\alpha,\quad \sqrt{ax^2+bx+c}=\beta;$$

from these equations we shall obtain *four* values of x.

When no sign is prefixed to a radical it is usually understood that it is to be taken as positive; hence, if α and β are both positive, all the four values of x satisfy the *original* equation. If however α or β is negative, the roots found from the resulting quadratic will satisfy the equation

$$ax^2+bx+c-p\sqrt{ax^2+bx+c}=q,$$

but not the original equation.

Example. Solve $x^2-5x+2\sqrt{x^2-5x+3}=12$.

Add 3 to each side; then

$$x^2-5x+3+2\sqrt{x^2-5x+3}=15.$$

Putting $\sqrt{x^2-5x+3}=y$, we obtain $y^2+2y-15=0$; whence $y=3$ or -5.

Thus $\sqrt{x^2-5x+3}=+3$, or $\sqrt{x^2-5x+3}=-5$.

Squaring, and solving the resulting quadratics, we obtain from the first $x=6$ or -1; and from the second $x=\dfrac{5\pm\sqrt{113}}{2}$. The first pair of values satisfies the given equation, but the second pair satisfies the equation

$$x^2-5x-2\sqrt{x^2-5x+3}=12.$$

131. Before clearing an equation of radicals it is advisable to examine whether any common factor can be removed by division.

Example. Solve $\sqrt{x^2-7ax+10a^2}-\sqrt{x^2+ax-6a^2}=x-2a.$

We have

$$\sqrt{(x-2a)(x-5a)}-\sqrt{(x-2a)(x+3a)}=x-2a.$$

The factor $\sqrt{x-2a}$ can now be removed from every term;

$$\therefore\ \sqrt{x-5a}-\sqrt{x+3a}=\sqrt{x-2a};$$

$$x-5a+x+3a-2\sqrt{(x-5a)(x+3a)}=x-2a;$$

$$x=2\sqrt{x^2-2ax-15a^2};$$

$$3x^2-8ax-60a^2=0;$$

$$(x-6a)(3x+10a)=0;$$

$$x=6a, \text{ or } -\frac{10a}{3}.$$

Also by equating to zero the factor $\sqrt{x-2a}$, we obtain $x=2a$.

On trial it will be found that $x=6a$ does not satisfy the equation: thus the roots are $-\frac{10a}{3}$ and $2a$.

The student may compare a similar question discussed in the *Elementary Algebra*, Art. 281.

132. The following artifice is sometimes useful.

Example. Solve $\sqrt{3x^2-4x+34}+\sqrt{3x^2-4x-11}=9$ ....(1).

We have *identically*

$$(3x^2-4x+34)-(3x^2-4x-11)=45 \text{ ....(2).}$$

Divide each member of (2) by the corresponding member of (1); thus

$$\sqrt{3x^2-4x+34}-\sqrt{3x^2-4x-11}=5 \text{ ....(3).}$$

Now (2) is an *identical equation* true for *all* values of x, whereas (1) is an equation which is true only for certain values of x; hence also equation (3) is only true for these values of x.

From (1) and (3) by addition

$$\sqrt{3x^2-4x+34}=7;$$

$$\text{whence } x=3, \text{ or } -\frac{5}{3}.$$

133. The solution of an equation of the form

$$ax^4 \pm bx^3 \pm cx^2 \pm bx + a = 0,$$

in which the coefficients of terms equidistant from the beginning and end are equal, can be made to depend on the solution of a quadratic. Equations of this type are known as *reciprocal equations*, and are so named because they are not altered when x is changed into its reciprocal $\frac{1}{x}$.

For a more complete discussion of reciprocal equations the student is referred to Arts. 568—570.

Example. Solve $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$.

Dividing by x^2 and rearranging,

$$12\left(x^2 + \frac{1}{x^2}\right) - 56\left(x + \frac{1}{x}\right) + 89 = 0.$$

Put $\qquad x + \frac{1}{x} = z$; then $x^2 + \frac{1}{x^2} = z^2 - 2$;

$$\therefore\ 12(z^2 - 2) - 56z + 89 = 0;$$

whence we obtain $\qquad z = \frac{5}{2}$, or $\frac{13}{6}$.

$$\therefore\ x + \frac{1}{x} = \frac{5}{2}, \text{ or } \frac{13}{6}.$$

By solving these equations we find that $x = 2, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}$.

134. The following equation though not *reciprocal* may be solved in a similar manner.

Example. Solve $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$.

We have $\qquad 6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x - \frac{1}{x}\right) + 12 = 0;$

whence $\qquad 6\left(x - \frac{1}{x}\right)^2 - 25\left(x - \frac{1}{x}\right) + 24 = 0;$

$$\therefore\ 2\left(x - \frac{1}{x}\right) - 3 = 0, \text{ or } 3\left(x - \frac{1}{x}\right) - 8 = 0;$$

whence we obtain $\qquad x = 2, -\frac{1}{2}, 3, -\frac{1}{3}$.

135. When one root of a quadratic equation is obvious by inspection, the other root may often be readily obtained by making use of the properties of the roots of quadratic equations proved in Art. 114.

Example. Solve $(1-a^2)(x+a)-2a(1-x^2)=0.$

This is a quadratic, one of whose roots is clearly a.

Also, since the equation may be written

$$2ax^2+(1-a^2)x-a(1+a^2)=0,$$

the product of the roots is $-\dfrac{1+a^2}{2}$; and therefore the other root is $-\dfrac{1+a^2}{2a}$.

EXAMPLES. X. a.

[*When any of the roots satisfy a modified form of the equation, the student should examine the particular arrangement of the signs of the radicals to which each solution applies.*]

Solve the following equations:

1. $x^{-2}-2x^{-1}=8.$
2. $9+x^{-4}=10x^{-2}.$
3. $2\sqrt{x}+2x^{-\frac{1}{2}}=5.$
4. $6x^{\frac{3}{4}}=7x^{\frac{1}{4}}-2x^{-\frac{1}{4}}.$
5. $x^{\frac{2}{n}}+6=5x^{\frac{1}{n}}.$
6. $3x^{\frac{1}{2n}}-x^{\frac{1}{n}}-2=0.$
7. $5\sqrt{\dfrac{3}{x}}+7\sqrt{\dfrac{x}{3}}=22\tfrac{2}{3}.$
8. $\sqrt{\dfrac{x}{1-x}}+\sqrt{\dfrac{1-x}{x}}=2\tfrac{1}{6}.$
9. $6\sqrt{x}=5x^{-\frac{1}{2}}-13.$
10. $1+8x^{\frac{6}{5}}+9\sqrt[5]{x^3}=0.$
11. $3^{2x}+9=10\,.\,3^x.$
12. $5(5^x+5^{-x})=26.$
13. $2^{2x+8}+1=32\,.\,2^x.$
14. $2^{2x+3}-57=65(2^x-1).$
15. $\sqrt{2^x}+\dfrac{1}{\sqrt{2^x}}=2.$
16. $\dfrac{3}{\sqrt{2x}}-\dfrac{\sqrt{2x}}{5}=5\tfrac{9}{10}.$
17. $(x-7)(x-3)(x+5)(x+1)=1680.$
18. $(x+9)(x-3)(x-7)(x+5)=385.$
19. $x(2x+1)(x-2)(2x-3)=63.$
20. $(2x-7)(x^2-9)(2x+5)=91.$
21. $x^2+2\sqrt{x^2+6x}=24-6x.$
22. $3x^2-4x+\sqrt{3x^2-4x-6}=18.$
23. $3x^2-7+3\sqrt{3x^2-16x+21}=16x.$
24. $8+9\sqrt{(3x-1)(x-2)}=3x^2-7x.$
25. $\dfrac{3x-2}{2}+\sqrt{2x^2-5x+3}=\dfrac{(x+1)^2}{3}.$

26. $7x - \dfrac{\sqrt{3x^2-8x+1}}{x} = \left(\dfrac{8}{\sqrt{x}} + \sqrt{x}\right)^2.$

27. $\sqrt{4x^2-7x-15} - \sqrt{x^2-3x} = \sqrt{x^2-9}.$

23. $\sqrt{2x^2-9x+4} + 3\sqrt{2x-1} = \sqrt{2x^2+21x-11}.$

29. $\sqrt{2x^2+5x-7} + \sqrt{3(x^2-7x+6)} - \sqrt{7x^2-6x-1} = 0.$

30. $\sqrt{a^2+2ax-3x^2} - \sqrt{a^2+ax-6x^2} = \sqrt{2a^2+3ax-9x^2}.$

31. $\sqrt{2x^2+5x-2} - \sqrt{2x^2+5x-9} = 1.$

32. $\sqrt{3x^2-2x+9} + \sqrt{3x^2-2x-4} = 13.$

33. $\sqrt{2x^2-7x+1} - \sqrt{2x^2-9x+4} = 1.$

34. $\sqrt{3x^2-7x-30} - \sqrt{2x^2-7x-5} = x-5.$

35. $x^4 + x^3 - 4x^2 + x + 1 = 0.$

36. $x^4 + \dfrac{8}{9}x^2 + 1 = 3x^3 + 3x.$

37. $x^4 + 1 - 3(x^3 + x) = 2x^2.$

38. $10(x^4+1) - 63x(x^2-1) + 52x^2 = 0.$

39. $\dfrac{x + \sqrt{12a-x}}{x - \sqrt{12a-x}} = \dfrac{\sqrt{a}+1}{\sqrt{a}-1}.$

40. $\dfrac{a+2x+\sqrt{a^2-4x^2}}{a+2x-\sqrt{a^2-4x^2}} = \dfrac{5x}{a}.$

41. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}} - \dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 8x\sqrt{x^2-3x+2}.$

42. $\sqrt{x^2+x} + \dfrac{\sqrt{x-1}}{\sqrt{x^3-x}} = \dfrac{5}{2}.$

43. $\dfrac{x^3+1}{x^2-1} = x + \sqrt{\dfrac{6}{x}}.$

44. $2^{x^2} : 2^{2x} = 8 : 1.$

45. $a^{2x}(a^2+1) = (a^{3x}+a^x)a.$

46. $\dfrac{8\sqrt{x}-5}{3x-7} = \dfrac{\sqrt{3x}-7}{x-5}.$

47. $\dfrac{18(7x-3)}{2x+1} = \dfrac{250\sqrt{2x+1}}{3\sqrt{7x-3}}.$

48. $(a+x)^{\frac{2}{3}} + 4(a-x)^{\frac{2}{3}} = 5(a^2-x^2)^{\frac{1}{3}}.$

49. $\sqrt{x^2+ax-1} - \sqrt{x^2+bx-1} = \sqrt{a} - \sqrt{b}.$

50. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}} + \dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 98.$

51. $x^4 - 2x^3 + x = 380.$

52. $27x^3 + 21x + 8 = 0.$

136. We shall now discuss some simultaneous equations of two unknown quantities.

Example 1. Solve $x+2+y+3+\sqrt{(x+2)(y+3)}=39$.

$$(x+2)^2+(y+3)^2+(x+2)(y+3)=741.$$

Put $x+2=u$, and $y+3=v$; then

$$u+v+\sqrt{uv}=39 \quad\text{.................(1)},$$

$$u^2+v^2+uv=741 \quad\text{.................(2)},$$

hence, from (1) and (2), we obtain by division,

$$u+v-\sqrt{uv}=19 \quad\text{.................(3)}.$$

From (1) and (3), $\quad u+v=29;$

and $\quad \sqrt{uv}=10,$

or $\quad uv=100;$

whence $\quad u=25, \text{ or } 4; \; v=4, \text{ or } 25;$

thus $\quad x=23, \text{ or } 2; \; y=1, \text{ or } 22.$

Example 2. Solve $\quad x^4+y^4=82 \quad\text{.................(1)},$

$$x-y=2 \quad\text{.................(2)}.$$

Put $\quad x=u+v, \text{ and } y=u-v;$

then from (2) we obtain $\quad v=1.$

Substituting in (1), $\quad (u+1)^4+(u-1)^4=82;$

$$\therefore\; 2(u^4+6u^2+1)=82;$$

$$u^4+6u^2-40=0;$$

whence $\quad u^2=4, \text{ or } -10;$

and $\quad u=\pm 2, \text{ or } \pm\sqrt{-10}.$

Thus $\quad x=3, \; -1, \; 1\pm\sqrt{-10};$

$$y=1, \; -3, \; -1\pm\sqrt{-10}.$$

Example 3. Solve $\quad \dfrac{2x+y}{3x-y}-\dfrac{x-y}{x+y}=2\tfrac{8}{15} \quad\text{.................(1)},$

$$7x+5y=29 \quad\text{.................(2)}.$$

From (1), $15(2x^2+3xy+y^2-3x^2+4xy-y^2)=38(3x^2+2xy-y^2);$

$$\therefore\; 129x^2-29xy-38y^2=0;$$

$$\therefore\; (3x-2y)(43x+19y)=0.$$

Hence $\quad 3x=2y \quad\text{.................(3)},$

or $\quad 43x=-19y \quad\text{.................(4)}.$

From (3), $$\frac{x}{2}=\frac{y}{3}=\frac{7x+5y}{29}$$

$$=1, \text{ by equation (2).}$$

$$\therefore\ x=2,\ y=3.$$

Again, from (4), $$\frac{x}{19}=\frac{y}{-43}=\frac{7x+5y}{-82}$$

$$=-\frac{29}{82}, \text{ by equation (2),}$$

$$\therefore\ x=-\frac{551}{82},\ y=\frac{1247}{82}.$$

Hence $$x=2,\ y=3;\ \text{or } x=-\frac{551}{82},\ y=\frac{1247}{82}.$$

Example 4. Solve $$4x^3+3x^2y+y^3=8,$$
$$2x^3-2x^2y+xy^2=1.$$

Put $y=mx$, and substitute in both equations. Thus

$$x^3(4+3m+m^3)=8 \quad \text{.............................(1).}$$

$$x^3(2-2m+m^2)=1 \quad \text{.............................(2).}$$

$$\therefore\ \frac{4+3m+m^3}{2-2m+m^2}=8;$$

$$m^3-8m^2+19m-12=0;$$

that is, $$(m-1)(m-3)(m-4)=0;$$

$$\therefore\ m=1, \text{ or } 3, \text{ or } 4.$$

(i) Take $m=1$, and substitute in either (1) or (2).

From (2), $$x^3=1;\ \therefore\ x=1;$$
and $$y=mx=x=1.$$

(ii) Take $m=3$, and substitute in (2);

thus $$5x^3=1;\ \therefore\ x=\sqrt[3]{\frac{1}{5}};$$

and $$y=mx=3x=3\sqrt[3]{\frac{1}{5}}.$$

(iii) Take $m=4$; we obtain

$$10x^3=1;\ \therefore\ x=\sqrt[3]{\frac{1}{10}};$$

and $$y=mx=4x=4\sqrt[3]{\frac{1}{10}}.$$

Hence the complete solution is

$$x = 1, \ \sqrt[3]{\frac{1}{5}}, \ \sqrt[3]{\frac{1}{10}}.$$

$$y = 1, \ 3\sqrt[3]{\frac{1}{5}}, \ 4\sqrt[3]{\frac{1}{10}}.$$

NOTE. The above method of solution may always be used when the equations are *of the same degree and homogeneous.*

Example 5. Solve $31x^2y^2 - 7y^4 - 112xy + 64 = 0$ ....(1),

$$x^2 - 7xy + 4y^2 + 8 = 0 \quad\text{.................................(2).}$$

From (2) we have $-8 = x^2 - 7xy + 4y^2$; and, substituting in (1),

$$31x^2y^2 - 7y^4 + 14xy\,(x^2 - 7xy + 4y^2) + (x^2 - 7xy + 4y^2)^2 = 0;$$

$$\therefore\ 31x^2y^2 - 7y^4 + (x^2 - 7xy + 4y^2)\,(14xy + x^2 - 7xy + 4y^2) = 0;$$

$$\therefore\ 31x^2y^2 - 7y^4 + (x^2 + 4y^2)^2 - (7xy)^2 = 0;$$

that is, $x^4 - 10x^2y^2 + 9y^4 = 0$..................................(3).

$$\therefore\ (x^2 - y^2)\,(x^2 - 9y^2) = 0;$$

hence $x = \pm y$, or $x = \pm 3y$.

Taking these cases in succession and substituting in (2), we obtain

$$x = y = \pm 2;$$

$$x = -y = \pm\sqrt{-\frac{2}{3}};$$

$$x = \pm 3, \ y = \pm 1;$$

$$x = \pm 3\sqrt{-\frac{4}{17}}, \ y = \mp\sqrt{-\frac{4}{17}}.$$

NOTE. It should be observed that equation (3) is *homogeneous.* The method here employed by which one equation is made homogeneous by a suitable combination with the other is a valuable artifice. It is especially useful in Analytical Geometry.

Example 6. Solve $(x+y)^{\frac{2}{3}} + 2\,(x-y)^{\frac{2}{3}} = 3\,(x^2 - y^2)^{\frac{1}{3}}$ ....(1).

$$3x - 2y = 13 \quad\text{...................................(2).}$$

Divide each term of (1) by $(x^2 - y^2)^{\frac{1}{3}}$, or $(x+y)^{\frac{1}{3}}\,(x-y)^{\frac{1}{3}}$;

$$\therefore\ \left(\frac{x+y}{x-y}\right)^{\frac{1}{3}} + 2\left(\frac{x-y}{x+y}\right)^{\frac{1}{3}} = 3.$$

This equation is a quadratic in $\left(\dfrac{x+y}{x-y}\right)^{\frac{1}{3}}$, from which we easily find,

$$\left(\frac{x+y}{x-y}\right)^{\frac{1}{3}}=2 \text{ or } 1; \quad \text{whence } \frac{x+y}{x-y}=8 \text{ or } 1;$$

$$\therefore\ 7x=9y, \text{ or } y=0.$$

Combining these equations with (2), we obtain

$$x=9,\ y=7; \text{ or } x=\frac{13}{3},\ y=0.$$

EXAMPLES. X. b.

Solve the following equations:

1. $3x-2y=7,\quad xy=20.$

2. $5x-y=3,\quad y^2-6x^2=25.$

3. $4x-3y=1,\quad 12xy+13y^2=25.$

4. $x^4+x^2y^2+y^4=931,\quad x^2-xy+y^2=19.$

5. $x^2+xy+y^2=84,\quad x-\sqrt{xy}+y=6.$

6. $x+\sqrt{xy}+y=65,\quad x^2+xy+y^2=2275.$

7. $x+y=7+\sqrt{xy},\quad x^2+y^2=133-xy.$

8. $3x^2-5y^2=7,\quad 3xy-4y^2=2.$

9. $5y^2-7x^2=17,\quad 5xy-6x^2=6.$

10. $3x^2+165=16xy,\quad 7xy+3y^2=132.$

11. $3x^2+xy+y^2=15,\quad 31xy-3x^2-5y^2=45.$

12. $x^2+y^2-3=3xy,\quad 2x^2-6+y^2=0.$

13. $x^4+y^4=706,\quad x+y=8.$

14. $x^4+y^4=272.\quad x-y=2.$

15. $x^5-y^5=992,\quad x-y=2.$

16. $x+\dfrac{4}{y}=1,\quad y+\dfrac{4}{x}=25.$

17. $\dfrac{x^2}{y}+\dfrac{y^2}{x}=\dfrac{9}{2},\quad \dfrac{3}{x+y}=1.$

18. $\dfrac{x}{2}+\dfrac{y}{5}=5.\quad \dfrac{2}{x}+\dfrac{5}{y}=\dfrac{5}{6}.$

19. $x+y=1072,\quad x^{\frac{1}{3}}+y^{\frac{1}{3}}=16.$

20. $xy^{\frac{1}{2}}+yx^{\frac{1}{2}}=20,\quad x^{\frac{3}{2}}+y^{\frac{3}{2}}=65.$

21. $x^{\frac{1}{2}}+y^{\frac{1}{2}}=5,\quad 6\left(x^{-\frac{1}{2}}+y^{-\frac{1}{2}}\right)=5.$

22. $\sqrt{x+y}+\sqrt{x-y}=4,$
$x^2-y^2=9.$

23. $y+\sqrt{x^2-1}=2,$
$\sqrt{x+1}-\sqrt{x-1}=\sqrt{y}.$

24. $\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{10}{3},$
$x+y=10.$

25. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}+\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}=\dfrac{17}{4},$
$x^2+y^2=706.$

26. $x^2+4y^2-15x=10(3y-8), \quad xy=6.$

27. $x^2y^2+400=41xy, \quad y^2=5xy-4x^2.$

28. $4x^2+5y=6+20xy-25y^2+2x, \quad 7x-11y=17.$

29. $9x^2+33x-12=12xy-4y^2+22y, \quad x^2-xy=18.$

30. $(x^2-y^2)(x-y)=16xy, \quad (x^4-y^4)(x^2-y^2)=640x^2y^2.$

31. $2x^2-xy+y^2=2y, \quad 2x^2+4xy=5y.$

32. $\dfrac{x^3+y^3}{(x+y)^2}+\dfrac{x^3-y^3}{(x-y)^2}=\dfrac{43x}{8}, \quad 5x-7y=4.$

33. $y(y^2-3xy-x^2)+24=0, \quad x(y^2-4xy+2x^2)+8=0.$

34. $3x^3-8xy^2+y^3+21=0, \quad x^2(y-x)=1.$

35. $y^2(4x^2-108)=x(x^3-9y^3), \quad 2x^2+9xy+y^2=108.$

36. $6x^4+x^2y^2+16=2x(12x+y^3), \quad x^2+xy-y^2=4.$

37. $x(a+x)=y(b+y), \quad ax+by=(x+y)^2.$

38. $xy+ab=2ax, \quad x^2y^2+a^2b^2=2b^2y^2.$

39. $\dfrac{x-a}{a^2}+\dfrac{y-b}{b^2}=\dfrac{1}{x-b}-\dfrac{1}{y-a}-\dfrac{1}{a-b}=0.$

40. $bx^3=10a^2bx+3a^3y, \quad ay^3=10ab^2y+3b^3x.$

41. $2a\left(\dfrac{x}{y}-\dfrac{y}{x}\right)+4a^2=4x^2+\dfrac{xy}{2a}-\dfrac{y^2}{a^2}=1.$

137. Equations involving three or more unknown quantities can only be solved in special cases. We shall here consider some of the most useful methods of solution.

Example 1. Solve
$$x+y+z=13 \quad \text{.......(1)},$$
$$x^2+y^2+z^2=65 \quad \text{.......(2)},$$
$$xy=10 \quad \text{.......(3)}.$$

From (2) and (3), $(x+y)^2+z^2=85.$

Put u for $x+y$; then this equation becomes
$$u^2+z^2=85.$$

Also from (1), $u+z=13$;

whence we obtain $u=7$ or 6; $z=6$ or 7.

Thus we have $\left.\begin{aligned} x+y&=7,\\ xy&=10\end{aligned}\right\}$ and $\left.\begin{aligned} x+y&=6,\\ xy&=10\end{aligned}\right\}$

Hence the solutions are

$$\left.\begin{aligned} x&=5, \text{ or } 2,\\ y&=2, \text{ or } 5,\\ z&=6;\end{aligned}\right\} \text{ or } \left.\begin{aligned} x&=3\pm\sqrt{-1},\\ y&=3\mp\sqrt{-1},\\ z&=7.\end{aligned}\right\}$$

Example 2. Solve
$$(x+y)(x+z)=30,$$
$$(y+z)(y+x)=15,$$
$$(z+x)(z+y)=18.$$

Write u, v, w for $y+z,\ z+x,\ x+y$ respectively; thus

$$vw=30,\quad wu=15,\quad uv=18 \quad\text{..............(1).}$$

Multiplying these equations together, we have

$$u^2v^2w^2=30\times15\times18=15^2\times6^2;$$
$$\therefore\ uvw=\pm90.$$

Combining this result with each of the equations in (1), we have

$$u=3,\ v=6,\ w=5;\ \text{or}\ u=-3,\ v=-6,\ w=-5;$$

$$\therefore\ \left.\begin{aligned} y+z&=3,\\ z+x&=6,\\ x+y&=5;\end{aligned}\right\} \text{ or } \left.\begin{aligned} y+z&=-3,\\ z+x&=-6,\\ x+y&=-5,\end{aligned}\right\}$$

whence $x=4,\ y=1,\ z=2$; or $x=-4,\ y=-1,\ z=-2$.

Example 3. Solve
$$y^2+yz+z^2=49 \quad\text{..............(1),}$$
$$z^2+zx+x^2=19 \quad\text{..............(2),}$$
$$x^2+xy+y^2=39 \quad\text{..............(3).}$$

Subtracting (2) from (1)

$$y^2-x^2+z(y-x)=30;$$

that is, $(y-x)(x+y+z)=30$ ....(4).

Similarly from (1) and (3)

$$(z-x)(x+y+z)=10 \quad\text{..............(5).}$$

Hence from (4) and (5), by division

$$\frac{y-x}{z-x}=3;$$

whence $y=3z-2x$.

Substituting in equation (3), we obtain

$$x^2 - 3xz + 3z^2 = 13.$$

From (2), $\qquad x^2 + \ xz + \ z^2 = 19.$

Solving these homogeneous equations as in Example 4, Art. 136, we obtain

$$x = \pm 2,\ z = \pm 3;\ \text{and therefore } y = \pm 5;$$

or $$x = \pm\frac{11}{\sqrt{7}},\ z = \pm\frac{1}{\sqrt{7}};\ \text{and therefore } y = \mp\frac{19}{\sqrt{7}}.$$

Example 4. Solve $x^2 - yz = a^2$, $y^2 - zx = b^2$, $z^2 - xy = c^2$.

Multiply the equations by y, z, x respectively and add; then

$$c^2x + a^2y + b^2z = 0 \qquad \text{(1)}.$$

Multiply the equations by z, x, y respectively and add; then

$$b^2x + c^2y + a^2z = 0 \qquad \text{(2)}.$$

From (1) and (2), by cross multiplication,

$$\frac{x}{a^4 - b^2c^2} = \frac{y}{b^4 - c^2a^2} = \frac{z}{c^4 - a^2b^2} = k \text{ suppose.}$$

Substitute in any one of the given equations; then

$$k^2(a^6 + b^6 + c^6 - 3a^2b^2c^2) = 1;$$

$$\therefore \frac{x}{a^4 - b^2c^2} = \frac{y}{b^4 - c^2a^2} = \frac{z}{c^4 - a^2b^2} = \pm\frac{1}{\sqrt{a^6 + b^6 + c^6 - 3a^2b^2c^2}}.$$

EXAMPLES. X. c.

Solve the following equations:

1. $9x + y - 8z = 0,$
   $4x - 8y + 7z = 0,$
   $yz + zx + xy = 47.$

2. $3x + y - 2z = 0,$
   $4x - y - 3z = 0,$
   $x^3 + y^3 + z^3 = 467.$

3. $x - y - z = 2,$
   $x^2 + y^2 - z^2 = 22,$
   $xy = 5.$

4. $x + 2y - z = 11,$
   $x^2 - 4y^2 + z^2 = 37,$
   $xz = 24.$

5. $x^2 + y^2 - z^2 = 21,$
   $3xz + 3yz - 2xy = 18,$
   $x + y - z = 5.$

6. $x^2 + xy + xz = 18,$
   $y^2 + yz + yx + 12 = 0,$
   $z^2 + zx + zy = 30.$

7. $x^2 + 2xy + 3xz = 50,$
   $2y^2 + 3yz + yx = 10,$
   $3z^2 + zx + 2zy = 10.$

8. $(y - z)(z + x) = 22,$
   $(z + x)(x - y) = 33,$
   $(x - y)(y - z) = 6.$

9. $x^2y^2z^2u=12,\ x^2y^2zu^2=8,\ x^2yz^2u^2=1,\ 3xy^2z^2u^2=4.$

10. $x^3y^2z=12,\ x^3yz^3=54,\ x^7y^3z^2=72.$

11. $xy+x+y=23,$
$xz+x+z=41,$
$yz+y+z=27.$

12. $2xy-4x+y=17,$
$3yz+y-6z=52,$
$6xz+3z+2x=29.$

13. $xz+y=7z,\ yz+x=8z,\ x+y+z=12.$

14. $x^3+y^3+z^3=a^3,\ x^2+y^2+z^2=a^2,\ x+y+z=a.$

15. $x^2+y^2+z^2=yz+zx+xy=a^2,\ 3x-y+z=a\sqrt{3}.$

16. $x^2+y^2+z^2=21a^2,\ yz+zx-xy=6a^2,\ 3x+y-2z=3a.$

Indeterminate Equations.

138. Suppose the following problem were proposed for solution:

A person spends £461 in buying horses and cows; if each horse costs £23 and each cow £16, how many of each does he buy?

Let x, y be the number of horses and cows respectively; then

$$23x+16y=461.$$

Here we have *one* equation involving *two* unknown quantities, and it is clear that by ascribing any value we please to x, we can obtain a corresponding value for y; thus it would appear at first sight that the problem admits of an infinite number of solutions. But it is clear from the nature of the question that x and y must be positive integers; and with this restriction, as we shall see later, the number of solutions is limited.

If the number of unknown quantities is greater than the number of independent equations, there will be an unlimited number of solutions, and the equations are said to be **indeterminate.** In the present section we shall only discuss the simplest kinds of indeterminate equations, confining our attention to *positive integral values* of the unknown quantities; it will be seen that this restriction enables us to express the solutions in a very simple form.

The general theory of indeterminate equations will be found in Chap. XXVI.

Example 1. Solve $7x+12y=220$ in positive integers.

Divide throughout by 7, the smaller coefficient; thus

$$x+y+\frac{5y}{7}=31+\frac{3}{7};$$

$$\therefore\ x+y+\frac{5y-3}{7}=31 \quad\text{.........................}\ (1)$$

Since x and y are to be integers, we must have

$$\frac{5y-3}{7}=\text{integer};$$

and therefore $$\frac{15y-9}{7}=\text{integer};$$

that is, $$2y-1+\frac{y-2}{7}=\text{integer};$$

and therefore $$\frac{y-2}{7}=\text{integer}=p \text{ suppose.}$$

$$\therefore\ y-2=7p,$$

or $$y=7p+2 \quad\text{..............................}(2).$$

Substituting this value of y in (1),

$$x+7p+2+5p+1=31;$$

that is, $$x=28-12p \quad\text{..}(3).$$

If in these results we give to p any integral value, we obtain corresponding integral values of x and y; but if $p>2$, we see from (3) that x is negative; and if p is a negative integer, y is negative. Thus the only *positive integral* values of x and y are obtained by putting $p=0, 1, 2$.

The complete solution may be exhibited as follows:

$$\left.\begin{array}{rrrr} p= & 0, & 1, & 2, \\ x= & 28, & 16, & 4, \\ y= & 2, & 9, & 16. \end{array}\right\}$$

NOTE. When we obtained $\frac{5y-3}{7}=\text{integer}$, we multiplied by 3 *in order to make the coefficient of y differ by unity from a multiple of* 7. A similar artifice should always be employed before introducing a symbol to denote the integer.

Example 2. Solve in positive integers, $14x-11y=29$ ....(1).

Divide by 11, the smaller coefficient; thus

$$x+\frac{3x}{11}-y=2+\frac{7}{11};$$

$$\therefore\ \frac{3x-7}{11}=2-x+y=\text{integer};$$

hence $$\frac{12x-28}{11} = \text{integer};$$

that is, $$x-2+\frac{x-6}{11} = \text{integer};$$

$$\therefore \frac{x-6}{11} = \text{integer} = p \text{ suppose};$$

and, from (1), $$\left.\begin{aligned} \therefore\ x &= 11p+6 \\ y &= 14p+5 \end{aligned}\right\}.$$

This is called the *general solution* of the equation, and by giving to p any positive integral value or zero, we obtain positive integral values of x and y; thus we have

$$\left.\begin{aligned} p &= 0,\ 1,\ 2,\ 3, \ldots\ldots\ldots\ldots \\ x &= 6,\ 17,\ 28,\ 39, \ldots\ldots\ldots\ldots \\ y &= 5,\ 19,\ 33,\ 47, \ldots\ldots\ldots\ldots \end{aligned}\right\},$$

the number of solutions being infinite.

Example 3. In how many ways can £5 be paid in half-crowns and florins?

Let x be the number of half-crowns, y the number of florins; then

$$5x+4y=200;$$

$$\therefore\ x+y+\frac{x}{4} = 50;$$

$$\therefore\ \frac{x}{4} = \text{integer} = p \text{ suppose};$$

$$\therefore\ x = 4p,$$

and $$y = 50-5p.$$

Solutions are obtained by ascribing to p the values 1, 2, 3, ...9; and therefore the number of ways is 9. If, however, the sum may be paid *either* in half-crowns *or* florins, p may also have the values 0 and 10. If $p=0$, then $x=0$, and the sum is paid entirely in florins; if $p=10$, then $y=0$, and the sum is paid entirely in half-crowns. Thus if zero values of x and y are admissible the number of ways is 11.

Example 4. The expenses of a party numbering 43 were £5. 14*s*. 6*d*.; if each man paid 5*s*., each woman 2*s*. 6*d*., and each child 1*s*., how many were there of each?

Let x, y, z denote the number of men, women, and children, respectively; then we have

$$x+y+z = 43 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$10x+5y+2z=229.$$

Eliminating z, we obtain $\quad 8x+3y=143.$

The general solution of this equation is

$$x = 3p+1,$$
$$y = 45-8p;$$

Hence by substituting in (1), we obtain

$$z = 5p - 3.$$

Here p cannot be negative or zero, but may have positive integral values from 1 to 5. Thus

$$\begin{aligned} p &= 1, \ 2, \ 3, \ 4, \ 5; \\ x &= 4, \ 7, \ 10, \ 13, \ 16; \\ y &= 37, \ 29, \ 21, \ 13, \ 5; \\ z &= 2, \ 7, \ 12, \ 17, \ 22. \end{aligned}$$

EXAMPLES. X. d.

Solve in positive integers:

1. $3x + 8y = 103$. **2.** $5x + 2y = 53$. **3.** $7x + 12y = 152$.

4. $13x + 11y = 414$. **5.** $23x + 25y = 915$. **6.** $41x + 47y = 2191$.

Find the general solution in positive integers, and the least values of x and y which satisfy the equations:

7. $5x - 7y = 3$. **8.** $6x - 13y = 1$. **9.** $8x - 21y = 33$.

10. $17y - 13x = 0$. **11.** $19y - 23x = 7$. **12.** $77y - 30x = 295$.

13. A farmer spends £752 in buying horses and cows; if each horse costs £37 and each cow £23, how many of each does he buy?

14. In how many ways can £5 be paid in shillings and sixpences, including zero solutions?

15. Divide 81 into two parts so that one may be a multiple of 8 and the other of 5.

16. What is the simplest way for a person who has only guineas to pay 10*s.* 6*d.* to another who has only half-crowns?

17. Find a number which being divided by 39 gives a remainder 16, and by 56 a remainder 27. How many such numbers are there?

18. What is the smallest number of florins that must be given to discharge a debt of £1. 6*s.* 6*d.*, if the change is to be paid in half-crowns only?

19. Divide 136 into two parts one of which when divided by 5 leaves remainder 2, and the other divided by 8 leaves remainder 3.

20. I buy 40 animals consisting of rams at £4, pigs at £2, and oxen at £17: if I spend £301, how many of each do I buy?

21. In my pocket I have 27 coins, which are sovereigns, half-crowns or shillings, and the amount I have is £5. 0*s.* 6*d.*; how many coins of each sort have I?

CHAPTER XI.

Permutations and Combinations.

139. Each of the *arrangements* which can be made by taking some or all of a number of things is called a **permutation.**

Each of the *groups* or *selections* which can be made by taking some or all of a number of things is called a **combination.**

Thus the *permutations* which can be made by taking the letters a, b, c, d two at a time are twelve in number, namely,

$$ab, \quad ac, \quad ad, \quad bc, \quad bd, \quad cd,$$
$$ba, \quad ca, \quad da, \quad cb, \quad db, \quad dc;$$

each of these presenting a different *arrangement* of two letters.

The *combinations* which can be made by taking the letters a, b, c, d two at a time are six in number: namely,

$$ab, \quad ac, \quad ad, \quad bc, \quad bd, \quad cd;$$

each of these presenting a different *selection* of two letters.

From this it appears that in forming *combinations* we are only concerned with the number of things each selection contains; whereas in forming *permutations* we have also to consider the order of the things which make up each arrangement; for instance, if from four letters a, b, c, d we make a selection of three, such as abc, this single combination admits of being arranged in the following ways:

$$abc, \quad acb, \quad bca, \quad bac, \quad cab, \quad cba,$$

and so gives rise to six different permutations.

140. Before discussing the general propositions of this chapter there is an important principle which we proceed to explain and illustrate by a few numerical examples.

If one operation can be performed in m *ways, and (when it has been performed in any one of these ways) a second operation can then be performed in* n *ways; the number of ways of performing the two operations will be* m × n.

If the first operation be performed in *any one* way, we can associate with this any of the n ways of performing the second operation: and thus we shall have n ways of performing the two operations without considering more than *one* way of performing the first; and so, corresponding to *each* of the m ways of performing the first operation, we shall have n ways of performing the two; hence altogether the number of ways in which the two operations can be performed is represented by the product $m \times n$.

Example 1. There are 10 steamers plying between Liverpool and Dublin; in how many ways can a man go from Liverpool to Dublin and return by a different steamer?

There are *ten* ways of making the first passage; and with each of these there is a choice of *nine* ways of returning (since the man is not to come back by the same steamer); hence the number of ways of making the two journeys is 10×9, or 90.

This principle may easily be extended to the case in which there are more than two operations each of which can be performed in a given number of ways.

Example 2. Three travellers arrive at a town where there are four hotels; in how many ways can they take up their quarters, each at a different hotel?

The first traveller has choice of four hotels, and when he has made his selection in any one way, the second traveller has a choice of three; therefore the first two can make their choice in 4×3 ways; and with any one such choice the third traveller can select his hotel in 2 ways; hence the required number of ways is $4 \times 3 \times 2$, or 24.

141. *To find the number of permutations of* n *dissimilar things taken* r *at a time.*

This is the same thing as finding the number of ways in which we can fill up r places when we have n different things at our disposal.

The first place may be filled up in n ways, for any one of the n things may be taken; when it has been filled up in any one of

these ways, the second place can then be filled up in $n-1$ ways; and since each way of filling up the first place can be associated with each way of filling up the second, the number of ways in which the first two places can be filled up is given by the product $n(n-1)$. And when the first two places have been filled up in any way, the third place can be filled up in $n-2$ ways. And reasoning as before, the number of ways in which three places can be filled up is $n(n-1)(n-2)$.

Proceeding thus, and noticing that a new factor is introduced with each new place filled up, and that at any stage the number of factors is the same as the number of places filled up, we shall have the number of ways in which r places can be filled up equal to

$$n(n-1)(n-2)\ldots\ldots \text{to } r \text{ factors};$$

and the r^{th} factor is

$$n-(r-1), \quad \text{or} \quad n-r+1.$$

Therefore the number of permutations of n things taken r at a time is

$$n(n-1)(n-2)\ldots\ldots(n-r+1).$$

Cor. The number of permutations of n things taken all at a time is

$$n(n-1)(n-2)\ldots\ldots \text{to } n \text{ factors},$$

or $$n(n-1)(n-2)\ldots\ldots 3 \,.\, 2 \,.\, 1.$$

It is usual to denote this product by the symbol $\underline{|n}$, which is read "factorial n." Also $n!$ is sometimes used for $\underline{|n}$.

142. We shall in future denote the number of permutations of n things taken r at a time by the symbol ${}^nP_r$, so that

$${}^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1);$$

also $${}^nP_n = \underline{|n}.$$

In working numerical examples it is useful to notice that the suffix in the symbol ${}^nP_r$ always denotes the number of factors in the formula we are using.

143. The number of permutations of n things taken r at a time may also be found in the following manner.

Let ${}^nP_r$ represent the number of permutations of n things taken r at a time.

Suppose we form all the permutations of n things taken $r-1$ at a time; the number of these will be ${}^nP_{r-1}$.

With *each of these* put one of the remaining $n-r+1$ things. Each time we do this we shall get one permutation of n things r at a time; and therefore the whole number of the permutations of n things r at a time is ${}^nP_{r-1} \times (n-r+1)$; that is,

$$ {}^nP_r = {}^nP_{r-1} \times (n-r+1). $$

By writing $r-1$ for r in this formula, we obtain

$$ {}^nP_{r-1} = {}^nP_{r-2} \times (n-r+2), $$

similarly,

$$ {}^nP_{r-2} = {}^nP_{r-3} \times (n-r+3), $$

$$ \cdots\cdots\cdots\cdots\cdots\cdots $$

$$ {}^nP_3 = {}^nP_2 \times (n-2), $$

$$ {}^nP_2 = {}^nP_1 \times (n-1), $$

$$ {}^nP_1 = n. $$

Multiply together the vertical columns and cancel like factors from each side, and we obtain

$$ {}^nP_r = n(n-1)(n-2) \ldots\ldots (n-r+1). $$

Example 1. Four persons enter a railway carriage in which there are six seats; in how many ways can they take their places?

The first person may seat himself in 6 ways; and then the second person in 5; the third in 4; and the fourth in 3; and since each of these ways may be associated with each of the others, the required answer is $6 \times 5 \times 4 \times 3$, or 360.

Example 2. How many different numbers can be formed by using six out of the nine digits 1, 2, 3,...9?

Here we have 9 different things and we have to find the number of permutations of them taken 6 at a time;

$$ \therefore \text{ the required result} = {}^9P_6 $$
$$ = 9 \times 8 \times 7 \times 6 \times 5 \times 4 $$
$$ = 60480. $$

144. *To find the number of combinations of* n *dissimilar things taken* r *at a time.*

Let ${}^nC_r$ denote the required number of combinations.

Then each of these combinations consists of a group of r dissimilar things which can be arranged among themselves in $|\underline{r}$ ways. [Art. 142.]

Hence ${}^nC_r \times |\underline{r}$ is equal to the number of *arrangements* of n things taken r at a time; that is,

$$
\begin{aligned}
{}^nC_r \times |\underline{r} &= {}^nP_r \\
&= n(n-1)(n-2)\ldots(n-r+1);
\end{aligned}
$$

$$\therefore\ {}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{|\underline{r}} \quad \ldots\ldots\ldots (1).$$

Cor. This formula for ${}^nC_r$ may also be written in a different form; for if we multiply the numerator and the denominator by $|\underline{n-r}$ we obtain

$$\frac{n(n-1)(n-2)\ldots(n-r+1) \times |\underline{n-r}}{|\underline{r}\,|\underline{n-r}}.$$

The numerator now consists of the product of all the natural numbers from n to 1;

$$\therefore\ {}^nC_r = \frac{|\underline{n}}{|\underline{r}\,|\underline{n-r}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

It will be convenient to remember both these expressions for ${}^nC_r$, using (1) in all cases where a numerical result is required, and (2) when it is sufficient to leave it in an algebraical shape.

Note. If in formula (2) we put $r = n$, we have

$${}^nC_n = \frac{|\underline{n}}{|\underline{n}\,|\underline{0}} = \frac{1}{|\underline{0}};$$

but ${}^nC_n = 1$, so that if the formula is to be true for $r = n$, the symbol $|\underline{0}$ must be considered as equivalent to 1.

Example. From 12 books in how many ways can a selection of 5 be made, (1) when one specified book is always included, (2) when one specified book is always excluded?

(1) Since the specified book is to be included in every selection, we have only to choose 4 out of the remaining 11.

$$
\begin{aligned}
\text{Hence the number of ways} &= {}^{11}C_4 \\
&= \frac{11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4} \\
&= 330.
\end{aligned}
$$

(2) Since the specified book is always to be excluded, we have to select the 5 books out of the remaining 11.

Hence the number of ways $= {}^{11}C_5$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4 \times 5}$$

$$= 462.$$

145. *The number of combinations of* n *things* r *at a time is equal to the number of combinations of* n *things* n − r *at a time.*

In making all the possible combinations of n things, to each group of r things we select, there is left a corresponding group of $n-r$ things; that is, the number of combinations of n things r at a time is the same as the number of combinations of n things $n-r$ at a time;

$$\therefore \; {}^nC_r = {}^nC_{n-r}.$$

The proposition may also be proved as follows:

$${}^nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\;\underline{|n-(n-r)}} \qquad \text{[Art. 144.]}$$

$$= \frac{\underline{|n}}{\underline{|n-r}\;\underline{|r}}$$

$$= {}^nC_r.$$

Such combinations are called *complementary*.

Note. Put $r = n$, then ${}^nC_0 = {}^nC_n = 1$.

The result we have just proved is useful in enabling us to abridge arithmetical work.

Example. Out of 14 men in how many ways can an eleven be chosen?

The required number $= {}^{14}C_{11}$

$$= {}^{14}C_3$$

$$= \frac{14 \times 13 \times 12}{1 \times 2 \times 3}$$

$$= 364.$$

If we had made use of the formula ${}^{14}C_{11}$, we should have had to reduce an expression whose numerator and denominator each contained 11 factors.

146. *To find the number of ways in which* m + n *things can be divided into two groups containing* m *and* n *things respectively.*

This is clearly equivalent to finding the number of combinations of $m+n$ things m at a time, for every time we select one group of m things we leave a group of n things behind.

Thus the required number $= \dfrac{(m+n)!}{m!\,n!}$.

NOTE. If $n=m$, the groups are equal, and in this case the number of *different* ways of subdivision is $\dfrac{(2m)!}{m!\,m!\,2!}$; for in any one way it is possible to interchange the two groups without obtaining a new distribution.

147. *To find the number of ways in which* m + n + p *things can be divided into three groups containing* m, n, p *things severally.*

First divide $m+n+p$ things into two groups containing m and $n+p$ things respectively: the number of ways in which this can be done is
$$\frac{(m+n+p)!}{m!\,(n+p)!}.$$

Then the number of ways in which the group of $n+p$ things can be divided into two groups containing n and p things respectively is
$$\frac{(n+p)!}{n!\,p!}.$$

Hence the number of ways in which the subdivision into three groups containing m, n, p things can be made is
$$\frac{(m+n+p)!}{m!\,(n+p)!} \times \frac{(n+p)!}{n!\,p!}, \text{ or } \frac{(m+n+p)!}{m!\,n!\,p!}.$$

NOTE. If we put $n=p=m$, we obtain $\dfrac{(3m)!}{m!\,m!\,m!}$; but this formula regards as different all the possible orders in which the three groups can occur in any one mode of subdivision. And since there are $3!$ such orders corresponding to each mode of subdivision, the number of *different* ways in which subdivision into three *equal* groups can be made is $\dfrac{(3m)!}{m!\,m!\,m!\,3!}$.

Example. The number of ways in which 15 recruits can be divided into three equal groups is $\dfrac{15!}{5!\,5!\,5!\,3!}$; and the number of ways in which they can be drafted into three different regiments, five into each, is $\dfrac{15!}{5!\,5!\,5!}$.

148. In the examples which follow it is important to notice that the formula for *permutations* should not be used until the suitable *selections* required by the question have been made.

Example 1. From 7 Englishmen and 4 Americans a committee of 6 is to be formed; in how many ways can this be done, (1) when the committee contains exactly 2 Americans, (2) at least 2 Americans?

(1) We have to choose 2 Americans and 4 Englishmen.

The number of ways in which the Americans can be chosen is ${}^4C_2$; and the number of ways in which the Englishmen can be chosen is ${}^7C_4$. Each of the first groups can be associated with each of the second; hence

the required number of ways $= {}^4C_2 \times {}^7C_4$

$$= \frac{\underline{|4}}{\underline{|2}\,\underline{|2}} \times \frac{\underline{|7}}{\underline{|4}\,\underline{|3}}$$

$$= \frac{\underline{|7}}{\underline{|2}\,\underline{|2}\,\underline{|3}} = 210.$$

(2) The committee may contain 2, 3, or 4 Americans.

We shall exhaust all the suitable combinations by forming all the groups containing 2 Americans and 4 Englishmen; then 3 Americans and 3 Englishmen; and lastly 4 Americans and 2 Englishmen.

The *sum* of the three results will give the answer. Hence the required number of ways $= {}^4C_2 \times {}^7C_4 + {}^4C_3 \times {}^7C_3 + {}^4C_4 \times {}^7C_2$

$$= \frac{\underline{|4}}{\underline{|2}\,\underline{|2}} \times \frac{\underline{|7}}{\underline{|4}\,\underline{|3}} + \frac{\underline{|4}}{\underline{|3}} \times \frac{\underline{|7}}{\underline{|3}\,\underline{|4}} + 1 \times \frac{\underline{|7}}{\underline{|2}\,\underline{|5}}$$

$$= 210 + 140 + 21 = 371.$$

In this Example we have only to make use of the suitable formulæ for *combinations*, for we are not concerned with the possible arrangements of the members of the committee among themselves.

Example 2. Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels?

The number of ways of choosing the three consonants is ${}^7C_3$, and the number of ways of choosing the 2 vowels is ${}^4C_2$; and since each of the first groups can be associated with each of the second, the number of combined groups, each containing 3 consonants and 2 vowels, is ${}^7C_3 \times {}^4C_2$.

Further, each of these groups contains 5 letters, which may be arranged among themselves in $\underline{|5}$ ways. Hence

the required number of words $= {}^7C_3 \times {}^4C_2 \times \underline{|5}$

$$= \frac{\underline{|7}}{\underline{|3}\,\underline{|4}} \times \frac{\underline{|4}}{\underline{|2}\,\underline{|2}} \times \underline{|5}$$

$$= 5 \times \underline{|7}$$

$$= 25200.$$

Example 3. How many words can be formed out of the letters *article*, so that the vowels occupy the even places?

Here we have to put the 3 vowels in 3 specified places, and the 4 consonants in the 4 remaining places; the first operation can be done in $\lfloor 3$ ways, and the second in $\lfloor 4$. Hence

the required number of words $= \lfloor 3 \times \lfloor 4$

$= 144.$

In this Example the formula for permutations is immediately applicable, because by the statement of the question there is but one way of choosing the vowels, and one way of choosing the consonants.

EXAMPLES XI. a.

1. In how many ways can a consonant and a vowel be chosen out of the letters of the word *courage?*

2. There are 8 candidates for a Classical, 7 for a Mathematical, and 4 for a Natural Science Scholarship. In how many ways can the Scholarships be awarded?

3. Find the value of ${}^8P_7$, ${}^{25}P_5$, ${}^{24}C_4$, ${}^{19}C_{14}$.

4. How many different arrangements can be made by taking 5 of the letters of the word *equation?*

5. If four times the number of permutations of n things 3 together is equal to five times the number of permutations of $n-1$ things 3 together, find n.

6. How many permutations can be made out of the letters of the word *triangle?* How many of these will begin with t and end with e?

7. How many different selections can be made by taking four of the digits 3, 4, 7, 5, 8, 1? How many different numbers can be formed with four of these digits?

8. If ${}^{2n}C_3 : {}^nC_2 = 44 : 3$, find n.

9. How many changes can be rung with a peal of 5 bells?

10. How many changes can be rung with a peal of 7 bells, the tenor always being last?

11. On how many nights may a watch of 4 men be drafted from a crew of 24, so that no two watches are identical? On how many of these would any one man be taken?

12. How many arrangements can be made out of the letters of the word *draught*, the vowels never being separated?

13. In a town council there are 25 councillors and 10 aldermen; how many committees can be formed each consisting of 5 councillors and 3 aldermen?

14. Out of the letters A, B, C, p, q, r how many arrangements can be made (1) beginning with a capital, (2) beginning and ending with a capital?

15. Find the number of combinations of 50 things 46 at a time.

16. If ${}^nC_{12} = {}^nC_8$, find ${}^nC_{17}$, ${}^{22}C_n$.

17. In how many ways can the letters of the word *vowels* be arranged, if the letters *oe* can only occupy odd places?

18. From 4 officers and 8 privates, in how many ways can 6 be chosen (1) to include exactly one officer, (2) to include at least one officer?

19. In how many ways can a party of 4 or more be selected from 10 persons?

20. If ${}^{18}C_r = {}^{18}C_{r+2}$, find ${}^rC_5$.

21. Out of 25 consonants and 5 vowels how many words can be formed each consisting of 2 consonants and 3 vowels?

22. In a library there are 20 Latin and 6 Greek books; in how many ways can a group of 5 consisting of 3 Latin and 2 Greek books be placed on a shelf?

23. In how many ways can 12 things be divided equally among 4 persons?

24. From 3 capitals, 5 consonants, and 4 vowels, how many words can be made, each containing 3 consonants and 2 vowels, and beginning with a capital?

25. At an election three districts are to be canvassed by 10, 15, and 20 men respectively. If 45 men volunteer, in how many ways can they be allotted to the different districts?

26. In how many ways can 4 Latin and 1 English book be placed on a shelf so that the English book is always in the middle, the selection being made from 7 Latin and 3 English books?

27. A boat is to be manned by eight men, of whom 2 can only row on bow side and 1 can only row on stroke side; in how many ways can the crew be arranged?

28. There are two works each of 3 volumes, and two works each of 2 volumes; in how many ways can the 10 books be placed on a shelf so that volumes of the same work are not separated?

29. In how many ways can 10 examination papers be arranged so that the best and worst papers never come together?

30. An eight-oared boat is to be manned by a crew chosen from 11 men, of whom 3 can steer but cannot row, and the rest can row but cannot steer. In how many ways can the crew be arranged, if two of the men can only row on bow side?

31. Prove that the number of ways in which p positive and n negative signs may be placed in a row so that no two negative signs shall be together is ${}^{p+1}C_n$.

32. If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, find r.

33. How many different signals can be made by hoisting 6 differently coloured flags one above the other, when any number of them may be hoisted at once?

34. If ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$, find r.

149. Hitherto, in the formulæ we have proved, the things have been regarded as *unlike*. Before considering cases in which some one or more sets of things may be *like*, it is necessary to point out exactly in what sense the words *like* and *unlike* are used. When we speak of things being *dissimilar*, *different*, *unlike*, we imply that the things are *visibly unlike*, so as to be easily distinguishable from each other. On the other hand we shall always use the term *like* things to denote such as are alike to the eye and cannot be distinguished from each other. For instance, in Ex. 2, Art. 148, the consonants and the vowels may be said each to consist of a group of things united by a common characteristic, and thus in a certain sense to be of the same kind; but they cannot be regarded as like things, because there is an individuality existing among the things of each group which makes them easily distinguishable from each other. Hence, in the final stage of the example we considered each group to consist of five *dissimilar* things and therefore capable of $\lfloor 5$ arrangements among themselves. [Art. **141** Cor.]

150. Suppose we have to find all the possible ways of arranging 12 books on a shelf, 5 of them being Latin, 4 English, and the remainder in different languages.

The books in each language may be regarded as belonging to one class, united by a common characteristic; but if they were distinguishable from each other, the number of permutations would be $\lfloor 12$, since for the purpose of arrangement among themselves they are essentially different.

If, however, the books in the same language are not distinguishable from each other, we should have to find the number of ways in which 12 things can be arranged among themselves, when 5 of them are exactly alike of one kind, and 4 exactly alike of a second kind: a problem which is not directly included in any of the cases we have previously considered.

151. *To find the number of ways in which* n *things may be arranged among themselves, taking them all at a time, when* p *of the things are exactly alike of one kind,* q *of them exactly alike of another kind,* r *of them exactly alike of a third kind, and the rest all different.*

Let there be n letters; suppose p of them to be a, q of them to be b, r of them to be c, and the rest to be unlike.

Let x be the required number of permutations; then if the p letters a were replaced by p unlike letters different from any of the rest, from *any one* of the x permutations, without altering the position of any of the remaining letters, we could form $\lfloor p$ new permutations. Hence if this change were made in each of the x permutations we should obtain $x \times \lfloor p$ permutations.

Similarly, if the q letters b were replaced by q unlike letters, the number of permutations would be

$$x \times \lfloor p \times \lfloor q.$$

In like manner, by replacing the r letters c by r unlike letters, we should finally obtain $x \times \lfloor p \times \lfloor q \times \lfloor r$ permutations.

But the things are now all different, and therefore admit of $\lfloor n$ permutations among themselves. Hence

$$x \times \lfloor p \times \lfloor q \times \lfloor r = \lfloor n;$$

that is,

$$x = \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r};$$

which is the required number of permutations.

Any case in which the things are not all different may be treated similarly.

Example 1. How many different permutations can be made out of the letters of the word *assassination* taken all together?

We have here 13 letters of which 4 are s, 3 are a, 2 are i, and 2 are n. Hence the number of permutations

$$= \frac{\lfloor 13}{\lfloor 4 \lfloor 3 \lfloor 2 \lfloor 2}$$

$$= 13 \,.\, 11 \,.\, 10 \,.\, 9 \,.\, 8 \,.\, 7 \,.\, 3 \,.\, 5$$

$$= 1001 \times 10800 = 10810800.$$

Example 2. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1, so that the odd digits always occupy the odd places?

The odd digits 1, 3, 3, 1 can be arranged in their four places in

$$\frac{\lfloor 4}{\lfloor 2 \lfloor 2} \text{ ways } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The even digits 2, 4, 2 can be arranged in their three places in

$$\frac{\lfloor 3}{\lfloor 2} \text{ ways } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Each of the ways in (1) can be associated with each of the ways in (2).

Hence the required number $= \dfrac{\lfloor 4}{\lfloor 2 \lfloor 2} \times \dfrac{\lfloor 3}{\lfloor 2} = 6 \times 3 = 18.$

152. *To find the number of permutations of* n *things* r *at a time, when each thing may be repeated once, twice,......up to* r *times in any arrangement.*

Here we have to consider the number of ways in which r places can be filled up when we have n different things at our disposal, each of the n things being used as often as we please in any arrangement.

The first place may be filled up in n ways, and, when it has been filled up in any one way, the second place may also be filled up in n ways, since we are not precluded from using the same thing again. Therefore the number of ways in which the first two places can be filled up is $n \times n$ or n^2. The third place can also be filled up in n ways, and therefore the first three places in n^3 ways.

Proceeding in this manner, and noticing that at any stage the index of n is always the same as the number of places filled up, we shall have the number of ways in which the r places can be filled up equal to n^r.

Example. In how many ways can 5 prizes be given away to 4 boys, when each boy is eligible for all the prizes?

Any one of the prizes can be given in 4 ways; and then any one of the remaining prizes can also be given in 4 ways, since it may be obtained by the boy who has already received a prize. Thus two prizes can be given away in 4^2 ways, three prizes in 4^3 ways, and so on. Hence the 5 prizes can be given away in 4^5, or 1024 ways.

153. *To find the total number of ways in which it is possible to make a selection by taking some or all of* n *things.*

Each thing may be dealt with in two ways, for it may either ~~be taken or left~~; and since either way of dealing with any one thing may be associated with either way of dealing with each one of the others, the number of ways of dealing with the n things is

$$2 \times 2 \times 2 \times 2 \ldots\ldots \text{to } n \text{ factors.}$$

But this includes the case in which all the things are left, therefore~~, rejecting this case, the~~ total number of ways is $2^n - 1$.

This is often spoken of as "the total number of combinations" of n things.

Example. A man has 6 friends; in how many ways may he invite one or more of them to dinner?

He has to select some or all of his 6 friends; and therefore the number of ways is $2^6 - 1$, or 63.

This result can be verified in the following manner.

The guests may be invited singly, in twos, threes,......; therefore the number of selections $= {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$

$$= 6 + 15 + 20 + 15 + 6 + 1 = 63.$$

154. *To find for what value of* r *the number of combinations of* n *things* r *at a time is greatest.*

Since $${}^nC_r = \frac{n(n-1)(n-2)\ldots\ldots(n-r+2)(n-r+1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots (r-1)\, r},$$

and $${}^nC_{r-1} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{1 \,.\, 2 \,.\, 3 \ldots\ldots (r-1)};$$

$$\therefore \quad {}^nC_r = {}^nC_{r-1} \times \frac{n-r+1}{r}.$$

The multiplying factor $\frac{n-r+1}{r}$ may be written $\frac{n+1}{r} - 1$, which shews that it decreases as r increases. Hence as r receives

the values 1, 2, 3...... in succession, ${}^nC_r$ is continually increased until $\dfrac{n+1}{r} - 1$ becomes equal to 1 or less than 1.

Now $$\frac{n+1}{r} - 1 > 1,$$

so long as $$\frac{n+1}{r} > 2;$$

that is, $$\frac{n+1}{2} > r.$$

We have to choose the greatest value of r consistent with this inequality.

(1) Let n be even, and equal to $2m$; then
$$\frac{n+1}{2} = \frac{2m+1}{2} = m + \frac{1}{2};$$

and for all values of r up to m inclusive this is greater than r. Hence by putting $r = m = \dfrac{n}{2}$, we find that the greatest number of combinations is ${}^nC_{\frac{n}{2}}$.

(2) Let n be odd, and equal to $2m+1$; then
$$\frac{n+1}{2} = \frac{2m+2}{2} = m+1;$$

and for all values of r up to m inclusive this is greater than r; but when $r = m+1$ the multiplying factor becomes equal to 1, and
$${}^nC_{m+1} = {}^nC_m; \text{ that is, } {}^nC_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}};$$

and therefore the number of combinations is greatest when the things are taken $\dfrac{n+1}{2}$, or $\dfrac{n-1}{2}$ at a time; the result being the same in the two cases.

155. The formula for the number of combinations of n things r at a time may be found without assuming the formula for the number of permutations.

Let ${}^nC_r$ denote the number of combinations of n things taken r at a time; and let the n things be denoted by the letters $a, b, c, d, \ldots\ldots$.

Take away a; then with the remaining letters we can form ${}^{n-1}C_{r-1}$ combinations of $n-1$ letters taken $r-1$ at a time. With each of these write a; thus we see that of the combinations of n things r at a time, the number of those which contain a is ${}^{n-1}C_{r-1}$; similarly the number of those which contain b is ${}^{n-1}C_{r-1}$; and so for each of the n letters.

Therefore $n \times {}^{n-1}C_{r-1}$ is equal to the number of combinations r at a time which contain a, together with those that contain b, those that contain c, and so on.

But by forming the combinations in this manner, each particular one will be repeated r times. For instance, if $r=3$, the combination abc will be found among those containing a, among those containing b, and among those containing c. Hence

$${}^{n}C_{r} = {}^{n-1}C_{r-1} \times \frac{n}{r}.$$

By writing $n-1$ and $r-1$ instead of n and r respectively,

$${}^{n-1}C_{r-1} = {}^{n-2}C_{r-2} \times \frac{n-1}{r-1}.$$

Similarly, $${}^{n-2}C_{r-2} = {}^{n-3}C_{r-3} \times \frac{n-2}{r-2},$$

....

$${}^{n-r+2}C_{2} = {}^{n-r+1}C_{1} \times \frac{n-r+2}{2};$$

and finally, $${}^{n-r+1}C_{1} = n-r+1.$$

Multiply together the vertical columns and cancel like factors from each side; thus

$${}^{n}C_{r} = \frac{n(n-1)(n-2)\,......\,(n-r+1)}{r(r-1)(r-2)\,......\,1}.$$

156. *To find the total number of ways in which it is possible to make a selection by taking some or all out of* p + q + r + ... *things, whereof* p *are alike of one kind,* q *alike of a second kind,* r *alike of a third kind; and so on.*

The p things may be disposed of in $p+1$ ways; for we may take 0, 1, 2, 3, ... p of them. Similarly the q things may be disposed of in $q+1$ ways; the r things in $r+1$ ways; and so on.

Hence the number of ways in which all the things may be disposed of is $(p+1)(q+1)(r+1)\ldots\ldots$.

But this includes the case in which none of the things are taken; therefore, rejecting this case, the total number of ways is

$$(p+1)(q+1)(r+1)\ldots\ldots-1.$$

157. A general formula expressing the number of permutations, or combinations, of n things taken r at a time, when the things are not all different, may be somewhat complicated; but a particular case may be solved in the following manner.

Example. Find the number of ways in which (1) a selection, (2) an arrangement, of four letters can be made from the letters of the word *proportion*.

There are 10 letters of six different sorts, namely o, o, o; p, p; r, r; t; i; n.

In finding groups of four these may be classified as follows:

(1) Three alike, one different.

(2) Two alike, two others alike.

(3) Two alike, the other two different.

(4) All four different.

(1) The selection can be made in 5 ways; for each of the five letters, p, r, t, i, n, can be taken with the single group of the three like letters o.

(2) The selection can be made in ${}^3C_2$ ways; for we have to choose two out of the three pairs o, o; p, p; r, r. This gives 3 selections.

(3) This selection can be made in 3×10 ways; for we select one of the 3 pairs, and then two from the remaining 5 letters. This gives 30 selections.

(4) This selection can be made in ${}^6C_4$ ways, as we have to take 4 different letters to choose from the six o, p, r, t, i, n. This gives 15 selections.

Thus the total number of selections is $5+3+30+15$; that is, 53.

In finding the different arrangements of 4 letters we have to permute in all possible ways each of the foregoing groups.

(1) gives rise to $5\times\dfrac{\underline{|4}}{\underline{|3}}$, or 20 arrangements.

(2) gives rise to $3\times\dfrac{\underline{|4}}{\underline{|2}\,\underline{|2}}$, or 18 arrangements.

(3) gives rise to $30\times\dfrac{\underline{|4}}{\underline{|2}}$, or 360 arrangements.

(4) gives rise to $15\times\underline{|4}$, or 360 arrangements.

Thus the total number of arrangements is $20+18+360+360$; that is, 758.

EXAMPLES. XI. b.

1. Find the number of arrangements that can be made out of the letters of the words

(1) *independence*, (2) *superstitious*,
(3) *institutions*.

2. In how many ways can 17 billiard balls be arranged, if 7 of them are black, 6 red, and 4 white?

3. A room is to be decorated with fourteen flags; if 2 of them are blue, 3 red, 2 white, 3 green, 2 yellow, and 2 purple, in how many ways can they be hung?

4. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?

5. Find the number of arrangements which can be made out of the letters of the word *algebra*, without altering the relative positions of vowels and consonants.

6. On three different days a man has to drive to a railway station, and he can choose from 5 conveyances; in how many ways can he make the three journeys?

7. I have counters of n different colours, red, white, blue,......; in how many ways can I make an arrangement consisting of r counters, supposing that there are at least r of each different colour?

8. In a steamer there are stalls for 12 animals, and there are cows, horses, and calves (not less than 12 of each) ready to be shipped; in how many ways can the shipload be made?

9. In how many ways can n things be given to p persons, when there is no restriction as to the number of things each may receive?

10. In how many ways can five things be divided between two persons?

11. How many different arrangements can be made out of the letters in the expression $a^3b^2c^4$ when written at full length?

12. A letter lock consists of three rings each marked with fifteen different letters; find in how many ways it is possible to make an unsuccessful attempt to open the lock.

13. Find the number of triangles which can be formed by joining three angular points of a quindecagon.

14. A library has a copies of one book, b copies of each of two books, c copies of each of three books, and single copies of d books. In how many ways can these books be distributed, if all are out at once?

15. How many numbers less than 10000 can be made with the eight digits 1, 2, 3, 0, 4, 5, 6, 7?

16. In how many ways can the following prizes be given away to a class of 20 boys: first and second Classical, first and second Mathematical, first Science, and first French?

17. A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest; what is the total number of signals that can be made?

18. In how many ways can 7 persons form a ring? In how many ways can 7 Englishmen and 7 Americans sit down at a round table, no two Americans being together?

19. In how many ways is it possible to draw a sum of money from a bag containing a sovereign, a half-sovereign, a crown, a florin, a shilling, a penny, and a farthing?

20. From 3 cocoa nuts, 4 apples, and 2 oranges, how many selections of fruit can be made, taking at least one of each kind?

21. Find the number of different ways of dividing mn things into n equal groups.

22. How many signals can be made by hoisting 4 flags of different colours one above the other, when any number of them may be hoisted at once? How many with 5 flags?

23. Find the number of permutations which can be formed out of the letters of the word *series* taken three together?

24. There are p points in a plane, no three of which are in the same straight line with the exception of q, which are all in the same straight line; find the number (1) of straight lines, (2) of triangles which result from joining them.

25. There are p points in space, no four of which are in the same plane with the exception of q, which are all in the same plane; find how many planes there are each containing three of the points.

26. There are n different books, and p copies of each; find the number of ways in which a selection can be made from them.

27. Find the number of selections and of arrangements that can be made by taking 4 letters from the word *expression*.

28. How many permutations of 4 letters can be made out of the letters of the word *examination*?

29. Find the sum of all numbers greater than 10000 formed by using the digits 1, 3, 5, 7, 9, no digit being repeated in any number.

30. Find the sum of all numbers greater than 10000 formed by using the digits 0, 2, 4, 6, 8, no digit being repeated in any number.

31. If of $p+q+r$ things p be alike, and q be alike, and the rest different, shew that the total number of combinations is

$$(p+1)(q+1)2^r-1.$$

32. Shew that the number of permutations which can be formed from $2n$ letters which are either a's or b's is greatest when the number of a's is equal to the number of b's.

33. If the $n+1$ numbers $a, b, c, d, \ldots\ldots$ be all different, and each of them a prime number, prove that the number of different factors of the expression $a^m bcd\ldots\ldots$ is $(m+1)2^n-1$.

CHAPTER XII.

Mathematical Induction.

158. Many important mathematical formulæ are not easily demonstrated by a direct mode of proof; in such cases we frequently find it convenient to employ a method of proof known as **mathematical induction**, which we shall now illustrate.

Example 1. Suppose it is required to prove that the sum of the cubes of the first n natural numbers is equal to $\left\{\frac{n(n+1)}{2}\right\}^2$.

We can easily see by trial that the statement is true in simple cases, such as when $n=1$, or 2, or 3; and from this we might be led to *conjecture* that the formula was true in all cases. Assume that it is true when n terms are taken; that is, suppose

$$1^3+2^3+3^3+\ldots\ldots \text{ to } n \text{ terms} = \left\{\frac{n(n+1)}{2}\right\}^2.$$

Add the $(n+1)^{\text{th}}$ term, that is, $(n+1)^3$ to each side; then

$$1^3+2^3+3^3+\ldots\ldots \text{ to } n+1 \text{ terms} = \left\{\frac{n(n+1)}{2}\right\}^2+(n+1)^3$$

$$=(n+1)^2\left(\frac{n^2}{4}+n+1\right)$$

$$=\frac{(n+1)^2(n^2+4n+4)}{4}$$

$$=\left\{\frac{(n+1)(n+2)}{2}\right\}^2;$$

which is *of the same form* as the result we assumed to be true for n terms. $n+1$ taking the place of n; in other words, if the result is true when we take a certain number of terms, whatever that number may be, it is true when we increase that number by one; but we see that it is true when 3 terms are taken; therefore it is true when 4 terms are taken; it is therefore true when 5 terms are taken; and so on. Thus the result is true universally.

Example 2. To determine the product of n binomial factors of the form $x+a$.

By actual multiplication we have

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+bc+ca)x+abc;$$

$$\begin{aligned}(x+a)(x+b)(x+c)(x+d)=x^4&+(a+b+c+d)x^3\\&+(ab+ac+ad+bc+bd+cd)x^2\\&+(abc+abd+acd+bcd)x+abcd.\end{aligned}$$

In these results we observe that the following laws hold:

1. The number of terms on the right is one more than the number of binomial factors on the left.

2. The index of x in the first term is the same as the number of binomial factors; and in each of the other terms the index is one less than that of the preceding term.

3. The coefficient of the first term is unity; the coefficient of the second term is the sum of the letters a, b, c, ...; the coefficient of the third term is the sum of the products of these letters taken two at a time; the coefficient of the fourth term is the sum of their products taken three at a time; and so on; the last term is the product of all the letters.

Assume that these laws hold in the case of $n-1$ factors; that is, suppose

$$(x+a)(x+b)\dots(x+h)=x^{n-1}+p_1x^{n-2}+p_2x^{n-3}+p_3x^{n-4}+\dots+p_{n-1},$$

where

$$\begin{aligned}p_1&=a+b+c+\dots h;\\p_2&=ab+ac+\dots+ah+bc+bd+\dots\dots;\\p_3&=abc+abd+\dots\dots;\\&\dots\dots\dots\dots\dots\dots\dots\dots\\p_{n-1}&=abc\dots h.\end{aligned}$$

Multiply both sides by another factor $x+k$; thus

$$\begin{aligned}&(x+a)(x+b)\dots(x+h)(x+k)\\&=x^n+(p_1+k)x^{n-1}+(p_2+p_1k)x^{n-2}+(p_3+p_2k)x^{n-3}+\dots+p_{n-1}k.\end{aligned}$$

Now $p_1+k=(a+b+c+\dots+h)+k$

$=$ sum of all the n letters a, b, c, ...k;

$p_2+p_1k=p_2+k(a+b+\dots+h)$

$=$ sum of the products taken two at a time of all the n letters a, b, c, ... k;

$p_3+p_2k=p_3+k(ab+ac+\dots+ah+bc+\dots)$

$=$ sum of the products taken three at a time of all the n letters a, b, c, ... k;

$p_{n-1}k=$ product of all the n letters a, b, c, ... k.

If therefore the laws hold when $n-1$ factors are multiplied together, they hold in the case of n factors. But we have seen that they hold in the case of 4 factors; therefore they hold for 5 factors; therefore also for 6 factors; and so on; thus they hold universally. Therefore

$$(x+a)(x+b)(x+c)\ldots(x+k)=x^n+S_1x^{n-1}+S_2x^{n-2}+S_3x^{n-3}+\ldots+S_n$$

where $S_1=$ the sum of all the n letters $a, b, c \ldots k$;

$S_2=$ the sum of the products taken two at a time of these n letters.

....

$S_n=$ the product of all the n letters.

159. Theorems relating to divisibility may often be established by induction.

Example. Shew that x^n-1 is divisible by $x-1$ for all positive integral values of n.

By division $$\frac{x^n-1}{x-1}=x^{n-1}+\frac{x^{n-1}-1}{x-1};$$

if therefore $x^{n-1}-1$ is divisible by $x-1$, then x^n-1 is also divisible by $x-1$. But x^2-1 is divisible by $x-1$; therefore x^3-1 is divisible by $x-1$; therefore x^4-1 is divisible by $x-1$, and so on; hence the proposition is established.

Other examples of the same kind will be found in the chapter on the *Theory of Numbers*.

160. From the foregoing examples it will be seen that the only theorems to which induction can be applied are those which admit of successive cases corresponding to the order of the natural numbers 1, 2, 3, ... n.

EXAMPLES. XII.

Prove by Induction:

1. $1+3+5+\ldots\ldots+(2n-1)=n^2$.

2. $1^2+2^2+3^2+\ldots\ldots+n^2=\frac{1}{6}n(n+1)(2n+1)$.

3. $2+2^2+2^3+\ldots\ldots+2^n=2(2^n-1)$.

4. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots$ to n terms $=\frac{n}{n+1}$.

5. Prove by Induction that x^n-y^n is divisible by $x+y$ when n is even.

CHAPTER XIII.

Binomial Theorem. Positive Integral Index.

161. It may be shewn by actual multiplication that

$$(x+a)(x+b)(x+c)(x+d)$$
$$= x^4 + (a+b+c+d)\,x^3 + (ab+ac+ad+bc+bd+cd)\,x^2$$
$$+ (abc+abd+acd+bcd)\,x + abcd \qquad\qquad (1).$$

We may, however, write down this result by inspection; for the complete product consists of the sum of a number of partial products each of which is formed by multiplying together four letters, *one* being taken from *each* of the four factors. If we examine the way in which the various partial products are formed, we see that

(1) the term x^4 is formed by taking the letter x out of *each* of the factors.

(2) the terms involving x^3 are formed by taking the letter x out of *any three* factors, in every way possible, and *one* of the letters a, b, c, d out of the remaining factor.

(3) the terms involving x^2 are formed by taking the letter x out of *any two* factors, in every way possible, and *two* of the letters a, b, c, d out of the remaining factors.

(4) the terms involving x are formed by taking the letter x out of *any one* factor, and *three* of the letters a, b, c, d out of the remaining factors.

(5) the term independent of x is the product of all the letters a, b, c, d.

Example 1. $\qquad (x-2)(x+3)(x-5)(x+9)$

$$= x^4 + (-2+3-5+9)\,x^3 + (-6+10-18-15+27-45)\,x^2$$
$$+ (30-54+90-135)\,x + 270$$
$$= x^4 + 5x^3 - 47x^2 - 69x + 270.$$

Example 2. Find the coefficient of x^3 in the product

$$(x-3)(x+5)(x-1)(x+2)(x-8).$$

The terms involving x^3 are formed by multiplying together the x in *any three* of the factors, and *two* of the numerical quantities out of the two remaining factors; hence the coefficient is equal to the sum of the products of the quantities -3, 5, -1, 2, -8 taken two at a time.

Thus the required coefficient

$$= -15+3-6+24-5+10-40-2+8-16$$
$$= -39.$$

162. If in equation (1) of the preceding article we suppose $b=c=d=a$, we obtain

$$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4.$$

The method here exemplified of deducing a particular case from a more general result is one of frequent occurrence in Mathematics; for it often happens that it is more easy to prove a general proposition than it is to prove a particular case of it.

We shall in the next article employ the same method to prove a formula known as the **Binomial Theorem**, by which any binomial of the form $x+a$ can be raised to any assigned positive integral power.

163. *To find the expansion of* $(x+a)^n$ *when* n *is a positive integer.*

Consider the expression

$$(x+a)(x+b)(x+c)\ \ldots\ldots\ (x+k),$$

the number of factors being n.

The expansion of this expression is the continued product of the n factors, $x+a$, $x+b$, $x+c$, ... $x+k$, and every term in the expansion is of n dimensions, being a product formed by multiplying together n letters, *one* taken from each of these n factors.

The highest power of x is x^n, and is formed by taking the letter x from each of the n factors.

The terms involving x^{n-1} are formed by taking the letter x from *any* $n-1$ of the factors, and *one* of the letters a, b, c, ... k from the remaining factor; thus the coefficient of x^{n-1} in the final product is the sum of the letters a, b, c, ... k; denote it by S_1.

The terms involving x^{n-2} are formed by taking the letter x from *any* $n-2$ of the factors, and *two* of the letters a, b, c, ... k from the two remaining factors; thus the coefficient of x^{n-2} in the final product is the sum of the products of the letters a, b, c, ... k taken two at a time; denote it by S_2.

And, generally, the terms involving x^{n-r} are formed by taking the letter x from *any* $n-r$ of the factors, and r of the letters $a, b, c, \ldots k$ from the r remaining factors; thus the coefficient of x^{n-r} in the final product is the sum of the products of the letters $a, b, c, \ldots k$ taken r at a time; denote it by S_r.

The last term in the product is $abc \ldots k$; denote it by S_n.

Hence $\qquad (x+a)(x+b)(x+c) \ldots\ldots (x+k)$

$$= x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots + S_r x^{n-r} + \ldots + S_{n-1} x + S_n.$$

In S_1 the *number of terms* is n; in S_2 *the number of terms* is the same as the number of combinations of n things 2 at a time; that is, ${}^nC_2$; in S_3 *the number of terms* is ${}^nC_3$; and so on.

Now suppose $b, c, \ldots k$, each equal to a; then S_1 becomes ${}^nC_1 a$; S_2 becomes ${}^nC_2 a^2$; S_3 becomes ${}^nC_3 a^3$; and so on; thus

$$(x+a)^n = x^n + {}^nC_1 ax^{n-1} + {}^nC_2 a^2 x^{n-2} + {}^nC_3 a^3 x^{n-3} + \ldots + {}^nC_n a^n;$$

substituting for ${}^nC_1$, ${}^nC_2, \ldots$ we obtain

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1\,.\,2} a^2 x^{n-2} + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} a^3 x^{n-3} + \ldots + a^n,$$

the series containing $n+1$ terms.

This is the *Binomial Theorem*, and the expression on the right is said to be **the expansion** of $(x+a)^n$.

164. The Binomial Theorem may also be proved as follows:

By induction we can find the product of the n factors $x+a$, $x+b$, $x+c, \ldots x+k$ as explained in Art. 158, Ex. 2; we can then deduce the expansion of $(x+a)^n$ as in Art. 163.

165. The coefficients in the expansion of $(x+a)^n$ are very conveniently expressed by the symbols ${}^nC_1$, ${}^nC_2$, ${}^nC_3, \ldots {}^nC_n$. We shall, however, sometimes further abbreviate them by omitting n, and writing $C_1, C_2, C_3, \ldots C_n$. With this notation we have

$$(x+a)^n = x^n + C_1 ax^{n-1} + C_2 a^2 x^{n-2} + C_3 a^3 x^{n-3} + \ldots + C_n a^n.$$

If we write $-a$ in the place of a, we obtain

$$(x-a)^n = x^n + C_1(-a)x^{n-1} + C_2(-a)^2 x^{n-2} + C_3(-a)^3 x^{n-3} + \ldots + C_n(-a)^n$$

$$= x^n - C_1 ax^{n-1} + C_2 a^2 x^{n-2} - C_3 a^3 x^{n-3} + \ldots + (-1)^n C_n a^n.$$

Thus the terms in the expansion of $(x+a)^n$ and $(x-a)^n$ are *numerically* the same, but in $(x-a)^n$ they are alternately positive and negative, and the last term is positive or negative according as n is even or odd.

Example 1. Find the expansion of $(x+y)^6$.

By the formula,

$$(x+y)^6 = x^6 + {}^6C_1 x^5 y + {}^6C_2 x^4 y^2 + {}^6C_3 x^3 y^3 + {}^6C_4 x^2 y^4 + {}^6C_5 x y^5 + {}^6C_6 y^6$$
$$= x^6 + 6x^5 y + 15x^4 y^2 + 20x^3 y^3 + 15x^2 y^4 + 6xy^5 + y^6,$$

on calculating the values of ${}^6C_1$, ${}^6C_2$, ${}^6C_3$, ...

Example 2. Find the expansion of $(a-2x)^7$.

$$(a-2x)^7 = a^7 - {}^7C_1 a^6 (2x) + {}^7C_2 a^5 (2x)^2 - {}^7C_3 a^4 (2x)^3 + \ldots\ldots \text{ to 8 terms.}$$

Now remembering that ${}^nC_r = {}^nC_{n-r}$, after calculating the coefficients up to ${}^7C_3$, the rest may be written down at once; for ${}^7C_4 = {}^7C_3$; ${}^7C_5 = {}^7C_2$; and so on. Hence

$$(a-2x)^7 = a^7 - 7a^6 (2x) + \frac{7 \cdot 6}{1 \cdot 2} a^5 (2x)^2 - \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} a^4 (2x)^3 + \ldots\ldots$$
$$= a^7 - 7a^6 (2x) + 21a^5 (2x)^2 - 35a^4 (2x)^3 + 35a^3 (2x)^4 - 21a^2 (2x)^5 + 7a (2x)^6 - (2x)^7$$
$$= a^7 - 14a^6 x + 84a^5 x^2 - 280a^4 x^3 + 560a^3 x^4 - 672a^2 x^5 + 448ax^6 - 128x^7.$$

Example 3. Find the value of

$$(a + \sqrt{a^2-1})^7 + (a - \sqrt{a^2-1})^7.$$

We have here the sum of two expansions whose terms are numerically the same; but in the second expansion the second, fourth, sixth, and eighth terms are negative, and therefore destroy the corresponding terms of the first expansion. Hence the value

$$= 2\{a^7 + 21a^5 (a^2-1) + 35a^3 (a^2-1)^2 + 7a (a^2-1)^3\}$$
$$= 2a (64a^6 - 112a^4 + 56a^2 - 7).$$

166. In the expansion of $(x+a)^n$, the coefficient of the second term is ${}^nC_1$; of the third term is ${}^nC_2$; of the fourth term is ${}^nC_3$; and so on; the suffix in each term being one less than the number of the term to which it applies; hence ${}^nC_r$ is the coefficient of the $(r+1)^{\text{th}}$ term. This is called the **general term**, because by giving to r different numerical values any of the coefficients may be found from ${}^nC_r$; and by giving to x and a their appropriate indices any assigned term may be obtained. Thus the $(r+1)^{\text{th}}$ term may be written

$${}^nC_r x^{n-r} a^r, \text{ or } \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r} x^{n-r} a^r.$$

In applying this formula to any particular case, it should be observed that *the index of* a *is the same as the suffix of* C, *and that the sum of the indices of* x *and* a *is* n.

Example 1. Find the fifth term of $(a+2x^3)^{17}$.

The required term $= {}^{17}C_4\, a^{13}\,(2x^3)^4$

$$= \frac{17 \cdot 16 \cdot 15 \cdot 14}{1 \cdot 2 \cdot 3 \cdot 4} \times 16a^{13}x^{12}$$

$$= 38080a^{13}\,x^{12}.$$

Example 2. Find the fourteenth term of $(3-a)^{15}$.

The required term $= {}^{15}C_{13}\,(3)^2\,(-a)^{13}$

$$= {}^{15}C_2 \times (-9a^{13}) \qquad \text{[Art. 145.]}$$

$$= -945a^{13}.$$

167. The simplest form of the binomial theorem is the expansion of $(1+x)^n$. This is obtained from the general formula of Art. 163, by writing 1 in the place of x, and x in the place of a. Thus

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + \ldots + {}^nC_n x^n$$

$$= 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \ldots\ldots + x^n;$$

the general term being

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\lfloor r} x^r.$$

The expansion of a binomial may always be made to depend upon the case in which the first term is unity; thus

$$(x+y)^n = \left\{x\left(1+\frac{y}{x}\right)\right\}^n$$

$$= x^n(1+z)^n, \text{ where } z = \frac{y}{x}.$$

Example 1. Find the coefficient of x^{16} in the expansion of $(x^2-2x)^{10}$.

We have $(x^2-2x)^{10} = x^{20}\left(1-\dfrac{2}{x}\right)^{10}$;

and, since x^{20} multiplies every term in the expansion of $\left(1-\dfrac{2}{x}\right)^{10}$, we have in this expansion to seek the coefficient of the term which contains $\dfrac{1}{x^4}$.

Hence the required coefficient $= {}^{10}C_4\,(-2)^4$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} \times 16$$

$$= 3360.$$

In some cases the following method is simpler.

Example 2. Find the coefficient of x^r in the expansion of $\left(x^2+\frac{1}{x^3}\right)^n$.

Suppose that x^r occurs in the $(p+1)^{\text{th}}$ term.

The $(p+1)^{\text{th}}$ term $\quad = {}^nC_p\,(x^2)^{n-p}\left(\dfrac{1}{x^3}\right)^p$

$$= {}^nC_p\,x^{2n-5p}.$$

But this term contains x^r, and therefore $2n-5p=r$, or $p=\dfrac{2n-r}{5}$.

Thus the required coefficient $= {}^nC_p = {}^nC_{\frac{2n-r}{5}}$

$$= \frac{\underline{|n}}{\underline{\left|\tfrac{1}{5}(2n-r)\right.}\;\underline{\left|\tfrac{1}{5}(3n+r)\right.}}.$$

Unless $\dfrac{2n-r}{5}$ is a positive integer there will be no term containing x^r in the expansion.

168. In Art. 163 we deduced the expansion of $(x+a)^n$ from the product of n factors $(x+a)(x+b)\ldots(x+k)$, and the method of proof there given is valuable in consequence of the wide generality of the results obtained. But the following shorter proof of the Binomial Theorem should be noticed.

It will be seen in Chap. XV. that a similar method is used to obtain the general term of the expansion of

$$(a+b+c+\ldots\ldots)^n.$$

169. *To prove the Binomial Theorem.*

The expansion of $(x+a)^n$ is the product of n factors, each equal to $x+a$, and every term in the expansion is of n dimensions, being a product formed by multiplying together n letters, *one* taken from each of the n factors. Thus each term involving $x^{n-r}a^r$ is obtained by taking a out of *any* r of the factors, and x out of the remaining $n-r$ factors. Therefore the number of terms which involve $x^{n-r}a^r$ must be equal to the number of ways in which r things can be selected out of n; that is, the coefficient of $x^{n-r}a^r$ is ${}^nC_r$, and by giving to r the values 0, 1, 2, 3, … n in succession we obtain the coefficients of all the terms. Hence

$$(x+a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots + {}^nC_rx^{n-r}a^r + \ldots + a^n,$$

since ${}^nC_0$ and ${}^nC_n$ are each equal to unity.

EXAMPLES. XIII. a.

Expand the following binomials:

1. $(x-3)^5$.
2. $(3x+2y)^4$.
3. $(2x-y)^5$.
4. $(1-3a^2)^6$.
5. $(x^2+x)^5$.
6. $(1-xy)^7$.
7. $\left(2-\dfrac{3x^2}{2}\right)^4$.
8. $\left(3a-\dfrac{2}{3}\right)^6$.
9. $\left(1+\dfrac{x}{2}\right)^7$.
10. $\left(\dfrac{2}{3}x-\dfrac{3}{2x}\right)^6$.
11. $\left(\dfrac{1}{2}+a\right)^8$.
12. $\left(1-\dfrac{1}{x}\right)^{10}$.

Write down and simplify:

13. The 4th term of $(x-5)^{13}$.
14. The 10th term of $(1-2x)^{12}$.
15. The 12th term of $(2x-1)^{13}$.
16. The 28th term of $(5x+8y)^{30}$.
17. The 4th term of $\left(\dfrac{a}{3}+9b\right)^{10}$.
18. The 5th term of $\left(2a-\dfrac{b}{3}\right)^8$.
19. The 7th term of $\left(\dfrac{4x}{5}-\dfrac{5}{2x}\right)^9$.
20. The 5th term of $\left(\dfrac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}-\dfrac{y^{\frac{5}{2}}}{b^{\frac{3}{2}}}\right)^8$.

Find the value of

21. $(x+\sqrt{2})^4+(x-\sqrt{2})^4$.
22. $(\sqrt{x^2-a^2}+x)^5-(\sqrt{x^2-a^2}-x)^5$.
23. $(\sqrt{2}+1)^6-(\sqrt{2}-1)^6$.
24. $(2-\sqrt{1-x})^6+(2+\sqrt{1-x})^6$.
25. Find the middle term of $\left(\dfrac{a}{x}+\dfrac{x}{a}\right)^{10}$.
26. Find the middle term of $\left(1-\dfrac{x^2}{2}\right)^{14}$.
27. Find the coefficient of x^{18} in $\left(x^2+\dfrac{3a}{x}\right)^{15}$.
28. Find the coefficient of x^{18} in $(ax^4-bx)^9$.
29. Find the coefficients of x^{32} and x^{-17} in $\left(x^4-\dfrac{1}{x^3}\right)^{15}$.
30. Find the two middle terms of $\left(3a-\dfrac{a^3}{6}\right)^9$.

31. Find the term independent of x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

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32. Find the 13th term of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$.

33. If x^r occurs in the expansion of $\left(x + \frac{1}{x}\right)^n$, find its coefficient.

34. Find the term independent of x in $\left(x - \frac{1}{x^2}\right)^{3n}$.

35. If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, prove that its coefficient is $\dfrac{\lfloor 2n}{\lfloor \frac{1}{3}(4n-p) \, \lfloor \frac{1}{3}(2n+p)}$.

170. *In the expansion of* $(1 + x)^n$ *the coefficients of terms equidistant from the beginning and end are equal.*

The coefficient of the $(r+1)^{\text{th}}$ term from the beginning is ${}^nC_r$.

The $(r+1)^{\text{th}}$ term from the end has $n + 1 - (r + 1)$, or $n - r$ terms before it; therefore counting from the beginning it is the $(n - r + 1)^{\text{th}}$ term, and its coefficient is ${}^nC_{n-r}$, which has been shewn to be equal to ${}^nC_r$. [Art. 145.] Hence the proposition follows.

171. *To find the greatest coefficient in the expansion of* $(1 + x)^n$.

The coefficient of the general term of $(1 + x)^n$ is ${}^nC_r$; and we have only to find for what value of r this is greatest.

By Art. 154, when n is even, the greatest coefficient is ${}^nC_{\frac{n}{2}}$; and when n is odd, it is ${}^nC_{\frac{n-1}{2}}$, or ${}^nC_{\frac{n+1}{2}}$; these two coefficients being equal.

172. *To find the greatest term in the expansion of* $(x + a)^n$.

We have
$$(x + a)^n = x^n \left(1 + \frac{a}{x}\right)^n;$$
therefore, since x^n multiplies every term in $\left(1 + \frac{a}{x}\right)^n$, it will be sufficient to find the greatest term in this latter expansion.

Let the r^{th} and $(r+1)^{\text{th}}$ be any two consecutive terms. The $(r+1)^{\text{th}}$ term is obtained by multiplying the r^{th} term by $\dfrac{n-r+1}{r} \cdot \dfrac{a}{x}$; that is, by $\left(\dfrac{n+1}{r} - 1\right)\dfrac{a}{x}$. [Art. 166.]

The factor $\dfrac{n+1}{r} - 1$ decreases as r increases; hence the $(r+1)^{\text{th}}$ term is not always greater than the r^{th} term, but only until $\left(\dfrac{n+1}{r} - 1\right)\dfrac{a}{x}$ becomes equal to 1, or less than 1.

Now $$\left(\frac{n+1}{r} - 1\right)\frac{a}{x} > 1,$$

so long as $$\frac{n+1}{r} - 1 > \frac{x}{a};$$

that is, $$\frac{n+1}{r} > \frac{x}{a} + 1,$$

or $$\frac{n+1}{\frac{x}{a}+1} > r \quad\text{.............................}\quad (1).$$

If $\dfrac{n+1}{\frac{x}{a}+1}$ be an integer, denote it by p; then if $r = p$ the multiplying factor becomes 1, and the $(p+1)^{\text{th}}$ term is equal to the p^{th}; and these are greater than any other term.

If $\dfrac{n+1}{\frac{x}{a}+1}$ be not an integer, denote its integral part by q; then the greatest value of r consistent with (1) is q; hence the $(q+1)^{\text{th}}$ term is the greatest.

Since we are only concerned with the *numerically greatest term*, the investigation will be the same for $(x-a)^n$; therefore in any numerical example it is unnecessary to consider the sign of the second term of the binomial. Also it will be found best to work each example independently of the general formula.

Example 1. If $x=\frac{1}{3}$, find the greatest term in the expansion of $(1+4x)^8$.

Denote the r^{th} and $(r+1)^{\text{th}}$ terms by T_r and T_{r+1} respectively; then

$$T_{r+1}=\frac{8-r+1}{r}\cdot 4x\times T_r$$

$$=\frac{9-r}{r}\times\frac{4}{3}\times T_r;$$

hence $$T_{r+1}>T_r,$$

so long as $$\frac{9-r}{r}\times\frac{4}{3}>1;$$

that is $$36-4r>3r,$$

or $$36>7r.$$

The greatest value of r consistent with this is 5; hence the greatest term is the sixth, and its value

$$={}^8C_5\times\left(\frac{4}{3}\right)^5={}^8C_3\times\left(\frac{4}{3}\right)^5=\frac{57344}{243}.$$

Example 2. Find the greatest term in the expansion of $(3-2x)^9$ when $x=1$.

$$(3-2x)^9=3^9\left(1-\frac{2x}{3}\right)^9;$$

thus it will be sufficient to consider the expansion of $\left(1-\frac{2x}{3}\right)^9$.

Here $$T_{r+1}=\frac{9-r+1}{r}\cdot\frac{2x}{3}\times T_r,\ \textit{numerically},$$

$$=\frac{10-r}{r}\times\frac{2}{3}\times T_r;$$

hence $$T_{r+1}>T_r,$$

so long as $$\frac{10-r}{r}\times\frac{2}{3}>1;$$

that is, $$20>5r.$$

Hence for all values of r up to 3, we have $T_{r+1}>T_r$; but if $r=4$, then $T_{r+1}=T_r$, and these are the greatest terms. Thus the 4^{th} and 5^{th} terms are numerically equal and greater than any other term, and their value

$$=3^9\times{}^9C_3\times\left(\frac{2}{3}\right)^3=3^6\times 84\times 8=489888.$$

173. *To find the sum of the coefficients in the expansion of* $(1+x)^n$.

In the identity $(1+x)^n = 1 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, put $x = 1$; thus

$$2^n = 1 + C_1 + C_2 + C_3 + \ldots + C_n$$
$$= \text{sum of the coefficients.}$$

COR. $\qquad C_1 + C_2 + C_3 + \ldots + C_n = 2^n - 1;$

that is "the total number of combinations of n things" is $2^n - 1$. [Art. 153.]

174. *To prove that in the expansion of* $(1+x)^n$, *the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms.*

In the identity $(1+x)^n = 1 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, put $x = -1$; thus

$$0 = 1 - C_1 + C_2 - C_3 + C_4 - C_5 + \ldots\ldots;$$
$$\therefore\ 1 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots$$
$$= \frac{1}{2}\,(\text{sum of all the coefficients})$$
$$= 2^{n-1}.$$

175. The Binomial Theorem may also be applied to expand expressions which contain more than two terms.

Example. Find the expansion of $(x^2 + 2x - 1)^3$.

Regarding $2x - 1$ as a single term, the expansion

$$= (x^2)^3 + 3\,(x^2)^2\,(2x-1) + 3x^2\,(2x-1)^2 + (2x-1)^3$$
$$= x^6 + 6x^5 + 9x^4 - 4x^3 - 9x^2 + 6x - 1, \text{ on reduction.}$$

176. The following example is instructive.

Example. If $\qquad (1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots\ldots + c_n x^n$,

find the value of $\qquad c_0 + 2c_1 + 3c_2 + 4c_3 + \ldots\ldots + (n+1)\,c_n \ldots\ldots\ldots\ldots\ldots\ldots(1),$

and $\qquad c_1^2 + 2c_2^2 + 3c_3^2 + \ldots\ldots + nc_n^2 \ldots\ldots\ldots\ldots\ldots\ldots(2).$

The series (1) $= (c_0 + c_1 + c_2 + \ldots\ldots + c_n) + (c_1 + 2c_2 + 3c_3 + \ldots\ldots + nc_n)$

$$= 2^n + n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{1\,.\,2} + \ldots\ldots + 1\right\}$$
$$= 2^n + n\,(1+1)^{n-1}$$
$$= 2^n + n\,.\,2^{n-1}.$$

To find the value of the series (2), we proceed thus:

$$c_1x + 2c_2x^2 + 3c_3x^3 + \ldots\ldots + nc_nx^n$$

$$= nx\left\{1 + (n-1)x + \frac{(n-1)(n-2)}{1 \cdot 2}x^2 + \ldots\ldots + x^{n-1}\right\}$$

$$= nx(1+x)^{n-1};$$

hence, by changing x into $\frac{1}{x}$, we have

$$\frac{c_1}{x} + \frac{2c_2}{x^2} + \frac{3c_3}{x^3} + \ldots\ldots + \frac{nc_n}{x^n} = \frac{n}{x}\left(1 + \frac{1}{x}\right)^{n-1} \quad \ldots\ldots\ldots\ldots (3).$$

Also

$$c_0 + c_1x + c_2x^2 + \ldots\ldots + c_nx^n = (1+x)^n \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

If we multiply together the two series on the left-hand sides of (3) and (4), we see that in the product the term independent of x is the series (2); hence

the series (2) $= \text{term independent of } x \text{ in } \frac{n}{x}(1+x)^n\left(1 + \frac{1}{x}\right)^{n-1}$

$$= \text{term independent of } x \text{ in } \frac{n}{x^n}(1+x)^{2n-1}$$

$$= \text{coefficient of } x^n \text{ in } n(1+x)^{2n-1}$$

$$= n \times {}^{2n-1}C_n$$

$$= \frac{|\underline{2n-1}}{|\underline{n-1}\;|\underline{n-1}}.$$

EXAMPLES. XIII. b.

In the following expansions find which is the greatest term:

1. $(x-y)^{30}$ when $x=11$, $y=4$.
2. $(2x-3y)^{28}$ when $x=9$, $y=4$.
3. $(2a+b)^{14}$ when $a=4$, $b=5$.
4. $(3+2x)^{15}$ when $x=\frac{5}{2}$.

In the following expansions find the value of the greatest term:

5. $(1+x)^n$ when $x=\frac{2}{3}$, $n=6$.
6. $(a+x)^n$ when $a=\frac{1}{2}$, $x=\frac{1}{3}$, $n=9$.

7. Shew that the coefficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1+x)^{2n-1}$.

8. If A be the sum of the odd terms and B the sum of the even terms in the expansion of $(x+a)^n$, prove that $A^2 - B^2 = (x^2 - a^2)^n$.

9. The 2^{nd}, 3^{rd}, 4^{th} terms in the expansion of $(x+y)^n$ are 240, 720, 1080 respectively; find x, y, n.

10. Find the expansion of $(1+2x-x^2)^4$.

11. Find the expansion of $(3x^2-2ax+3a^2)^3$.

12. Find the r^{th} term from the end in $(x+a)^n$.

13. Find the $(p+2)^{\text{th}}$ term from the end in $\left(x-\dfrac{1}{x}\right)^{2n+1}$.

14. In the expansion of $(1+x)^{43}$ the coefficients of the $(2r+1)^{\text{th}}$ and the $(r+2)^{\text{th}}$ terms are equal; find r.

15. Find the relation between r and n in order that the coefficients of the $3r^{\text{th}}$ and $(r+2)^{\text{th}}$ terms of $(1+x)^{2n}$ may be equal.

16. Shew that the middle term in the expansion of $(1+x)^{2n}$ is

$$\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{\lfloor n} 2^n x^n.$$

If $c_0, c_1, c_2, \ldots c_n$ denote the coefficients in the expansion of $(1+x)^n$, prove that

17. $c_1 + 2c_2 + 3c_3 + \ldots\ldots + nc_n = n \cdot 2^{n-1}$.

18. $c_0 + \dfrac{c_1}{2} + \dfrac{c_2}{3} + \ldots\ldots + \dfrac{c_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$.

19. $\dfrac{c_1}{c_0} + \dfrac{2c_2}{c_1} + \dfrac{3c_3}{c_2} + \ldots\ldots + \dfrac{nc_n}{c_{n-1}} = \dfrac{n(n+1)}{2}$.

20. $(c_0+c_1)(c_1+c_2)\ldots\ldots(c_{n-1}+c_n) = \dfrac{c_1 c_2 \ldots\ldots c_n (n+1)^n}{\lfloor n}$.

21. $2c_0 + \dfrac{2^2 c_1}{2} + \dfrac{2^3 c_2}{3} + \dfrac{2^4 c_3}{4} + \ldots\ldots + \dfrac{2^{n+1} c_n}{n+1} = \dfrac{3^{n+1}-1}{n+1}$.

22. $c_0^2 + c_1^2 + c_2^2 + \ldots\ldots + c_n^2 = \dfrac{\lfloor 2n}{\lfloor n \lfloor n}$.

23. $c_0 c_r + c_1 c_{r+1} + c_2 c_{r+2} + \ldots\ldots + c_{n-r} c_n = \dfrac{\lfloor 2n}{\lfloor n-r \lfloor n+r}$.

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CHAPTER XIV.

Binomial Theorem. Any Index.

177. In the last chapter we investigated the Binomial Theorem when the index was any positive integer; we shall now consider whether the formulae there obtained hold in the case of negative and fractional values of the index.

Since, by Art. 167, every binomial may be reduced to one common type, it will be sufficient to confine our attention to binomials of the form $(1+x)^n$.

By actual evolution, we have

$$(1+x)^{\frac{1}{2}} = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots\dots;$$

and by actual division,

$$(1-x)^{-2} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots\dots;$$

[Compare Ex. 1, Art. 60.]

and in each of these series the number of terms is unlimited.

In these cases we have by independent processes obtained an expansion for each of the expressions $(1+x)^{\frac{1}{2}}$ and $(1+x)^{-2}$. We shall presently prove that they are only particular cases of the general formula for the expansion of $(1+x)^n$, where n is any rational quantity.

This formula was discovered by Newton.

10 178. Suppose we have two expressions arranged in ascending powers of x, such as

$$1 + mx + \frac{m(m-1)}{1 \,.\, 2}x^2 + \frac{m(m-1)(m-2)}{1 \,.\, 2 \,.\, 3}x^3 + \dots\dots\dots (1),$$

and $$1 + nx + \frac{n(n-1)}{1 \,.\, 2}x^2 + \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3}x^3 + \dots\dots\dots\dots (2).$$

The product of these two expressions will be a series in ascending powers of x; denote it by

$$1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots\dots ;$$

then it is clear that A, B, C, ... are functions of m and n, and therefore the actual values of A, B, C, ... in any particular case will depend upon the values of m and n in that case. But the way in which the coefficients of the powers of x in (1) and (2) combine to give A, B, C, ... is quite independent of m and n; in other words, *whatever values* m *and* n *may have*, A, B, C, ... *preserve the same invariable form.* If therefore we can determine the form of A, B, C, ... for any value of m and n, we conclude that A, B, C, ... will have the same form *for all values* of m and n.

The principle here explained is often referred to as an example of "the permanence of equivalent forms;" in the present case we have only to recognise the fact that *in any algebraical product* the *form* of the result will be the same whether the quantities involved are whole numbers, or fractions; positive, or negative.

We shall make use of this principle in the general proof of the Binomial Theorem for any index. The proof which we give is due to Euler.

179. *To prove the Binomial Theorem when the index is a positive fraction.*

Whatever be the value of m, *positive or negative, integral or fractional*, let the symbol $f(m)$ stand for the series

$$1 + mx + \frac{m(m-1)}{1 \cdot 2} x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} x^3 + \dots ;$$

then $f(n)$ will stand for the series

$$1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \dots .$$

If we multiply these two series together the product will be another series in ascending powers of x, whose *coefficients will be unaltered in form whatever* m *and* n *may be.*

To determine this *invariable form of the product* we may give to m and n any values that are most convenient; for this purpose suppose that m and n are positive integers. In this case $f(m)$ is the expanded form of $(1+x)^m$, and $f(n)$ is the expanded form of $(1+x)^n$; and therefore

$$f(m) \times f(n) = (1+x)^m \times (1+x)^n = (1+x)^{m+n},$$

but when m and n are positive integers the expansion of $(1+x)^{m+n}$ is

$$1 + (m+n)x + \frac{(m+n)(m+n-1)}{1 \cdot 2}x^2 + \ldots\ldots .$$

This then is the *form* of the product of $f(m) \times f(n)$ *in all cases*, whatever the values of m and n may be; and in agreement with our previous notation it may be denoted by $f(m+n)$; therefore *for all values of* m *and* n

$$f(m) \times f(n) = f(m+n).$$

Also
$$f(m) \times f(n) \times f(p) = f(m+n) \times f(p)$$
$$= f(m+n+p), \text{ similarly.}$$

Proceeding in this way we may shew that

$$f(m) \times f(n) \times f(p) \ldots \text{to } k \text{ factors} = f(m+n+p+\ldots \text{to } k \text{ terms}).$$

Let each of these quantities m, n, p, ... be equal to $\frac{h}{k}$, where h and k are positive integers;

$$\therefore \quad \left\{f\left(\frac{h}{k}\right)\right\}^k = f(h);$$

but since h is a positive integer, $f(h) = (1+x)^h$;

$$\therefore \quad (1+x)^h = \left\{f\left(\frac{h}{k}\right)\right\}^k;$$

$$\therefore \quad (1+x)^{\frac{h}{k}} = f\left(\frac{h}{k}\right);$$

but $f\left(\frac{h}{k}\right)$ stands for the series

$$1 + \frac{h}{k}x + \frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1 \cdot 2}x^2 + \ldots\ldots;$$

$$\therefore \quad (1+x)^{\frac{h}{k}} = 1 + \frac{h}{k}x + \frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1 \cdot 2}x^2 + \ldots\ldots,$$

which proves the Binomial Theorem for any positive fractional index.

180. *To prove the Binomial Theorem when the index is any negative quantity.*

It has been proved that

$$f(m) \times f(n) = f(m+n)$$

for all values of m and n. Replacing m by $-n$ (where n is positive), we have

$$f(-n) \times f(n) = f(-n+n)$$
$$= f(0)$$
$$= 1,$$

since all terms of the series except the first vanish;

$$\therefore \quad \frac{1}{f(n)} = f(-n);$$

but $f(n) = (1+x)^n$, for *any positive* value of n;

$$\therefore \quad \frac{1}{(1+x)^n} = f(-n),$$

or
$$(1+x)^{-n} = f(-n).$$

But $f(-n)$ stands for the series

$$1 + (-n)x + \frac{(-n)(-n-1)}{1 \cdot 2}x^2 + \ldots\ldots;$$

$$\therefore \quad (1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{1 \cdot 2}x^2 + \ldots\ldots;$$

which proves the Binomial Theorem for any negative index. Hence the theorem is completely established.

181. The proof contained in the two preceding articles may not appear wholly satisfactory, and will probably present some difficulties to the student. There is only one point to which we shall now refer.

In the expression for $f(m)$ the number of terms is finite when m is a positive integer, and unlimited in all other cases. See Art. 182. It is therefore necessary to enquire in what sense we

are to regard the statement that $f(m) \times f(n) = f(m+n)$. It will be seen in Chapter XXI., that when $x<1$, each of the series $f(m)$, $f(n)$, $f(m+n)$ is *convergent*, and $f(m+n)$ is the true arithmetical equivalent of $f(m) \times f(n)$. But when $x>1$, all these series are *divergent*, and we can only assert that if we multiply the series denoted by $f(m)$ by the series denoted by $f(n)$, the first r terms of the product will agree with the first r terms of $f(m+n)$, whatever finite value r may have. [See Art. 308.]

Example 1. Expand $(1-x)^{\frac{3}{2}}$ to four terms.

$$(1-x)^{\frac{3}{2}} = 1 + \frac{3}{2}(-x) + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{1 \, . \, 2}(-x)^2 + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{1 \, . \, 2 \, . \, 3}(-x)^3 + \ldots\ldots$$

$$= 1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \ldots\ldots$$

Example 2. Expand $(2+3x)^{-4}$ to four terms.

$$(2+3x)^{-4} = 2^{-4}\left(1+\frac{3x}{2}\right)^{-4}$$

$$= \frac{1}{2^4}\left[1 + (-4)\left(\frac{3x}{2}\right) + \frac{(-4)(-5)}{1 \, . \, 2}\left(\frac{3x}{2}\right)^2 + \frac{(-4)(-5)(-6)}{1 \, . \, 2 \, . \, 3}\left(\frac{3x}{2}\right)^3 + \ldots\ldots\right]$$

$$= \frac{1}{16}\left(1 - 6x + \frac{45}{2}x^2 - \frac{135}{2}x^3 + \ldots\ldots\right).$$

182. In finding the general term we must now use the formula

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\underline{|r}}x^r$$

written in full; for the symbol ${}^nC_r$ can no longer be employed when n is fractional or negative.

Also the coefficient of the general term can ~~never vanish unless~~ one of the factors of its numerator ~~is zero~~; the series will therefore stop at the ~~r^{th} term,~~ when $n-r+1$ ~~is zero~~; that is, when $r=n+1$; but since r is a positive integer this equality can never hold except when the index n is positive and integral. Thus the expansion by the Binomial Theorem extends to $n+1$ terms when n is a positive integer, and to an infinite number of terms in all other cases.

Example 1. Find the general term in the expansion of $(1+x)^{\frac{1}{2}}$.

$$\text{The } (r+1)^{\text{th}} \text{ term} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\ldots\left(\frac{1}{2}-r+1\right)}{\underline{|r}}x^r$$

$$= \frac{1(-1)(-3)(-5)\ldots\ldots(-2r+3)}{2^r\underline{|r}}x^r.$$

The number of factors in the numerator is r, and $r-1$ of these are negative; therefore, by taking -1 out of each of these negative factors, we may write the above expression

$$(-1)^{r-1}\frac{1\,.\,3\,.\,5\ldots\ldots(2r-3)}{2^r\,\underline{|r}}x^r.$$

Example 2. Find the general term in the expansion of $(1-nx)^{\frac{1}{n}}$.

$$\text{The } (r+1)^{\text{th}} \text{ term} = \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\ldots\ldots\left(\frac{1}{n}-r+1\right)}{\underline{|r}}(-nx)^r$$

$$= \frac{1(1-n)(1-2n)\ldots\ldots(1-\overline{r-1}\,.\,n)}{n^r\,\underline{|r}}(-1)^r n^r x^r$$

$$= (-1)^r\frac{1(1-n)(1-2n)\ldots\ldots(1-\overline{r-1}\,.\,n)}{\underline{|r}}x^r$$

$$= (-1)^r(-1)^{r-1}\frac{(n-1)(2n-1)\ldots\ldots(\overline{r-1}\,.\,n-1)}{\underline{|r}}x^r$$

$$= -\frac{(n-1)(2n-1)\ldots\ldots(\overline{r-1}\,.\,n-1)}{\underline{|r}}x^r,$$

since $$(-1)^r(-1)^{r-1} = (-1)^{2r-1} = -1.$$

Example 3. Find the general term in the expansion of $(1-x)^{-3}$.

$$\text{The } (r+1)^{\text{th}} \text{ term} = \frac{(-3)(-4)(-5)\ldots\ldots(-3-r+1)}{\underline{|r}}(-x)^r$$

$$= (-1)^r\frac{3\,.\,4\,.\,5\ldots\ldots(r+2)}{\underline{|r}}(-1)^r x^r$$

$$= (-1)^{2r}\frac{3\,.\,4\,.\,5\ldots\ldots(r+2)}{1\,.\,2\,.\,3\ldots\ldots r}x^r$$

$$= \frac{(r+1)(r+2)}{1\,.\,2}x^r,$$

by removing like factors from the numerator and denominator.

EXAMPLES. XIV. a.

Expand to 4 terms the following expressions:

1. $(1+x)^{\frac{1}{2}}$.
2. $(1+x)^{\frac{3}{2}}$.
3. $(1-x)^{\frac{2}{5}}$.
4. $(1+x^2)^{-2}$.
5. $(1-3x)^{\frac{1}{3}}$.
6. $(1-3x)^{-\frac{1}{3}}$.
7. $(1+2x)^{-\frac{1}{2}}$.
8. $\left(1+\frac{x}{3}\right)^{-3}$.
9. $\left(1+\frac{2x}{3}\right)^{\frac{3}{2}}$.
10. $\left(1+\frac{1}{2}a\right)^{-4}$.
11. $(2+x)^{-3}$.
12. $(9+2x)^{\frac{1}{2}}$.
13. $(8+12a)^{\frac{2}{3}}$.
14. $(9-6x)^{-\frac{3}{2}}$.
15. $(4a-8x)^{-\frac{1}{2}}$.

Write down and simplify:

16. The 8^{th} term of $(1+2x)^{-\frac{1}{2}}$.
17. The 11^{th} term of $(1-2x^3)^{\frac{11}{2}}$.
18. The 10^{th} term of $(1+3a^2)^{\frac{16}{3}}$.
19. The 5^{th} term of $(3a-2b)^{-1}$.
20. The $(r+1)^{\text{th}}$ term of $(1-x)^{-2}$.
21. The $(r+1)^{\text{th}}$ term of $(1-x)^{-4}$.
22. The $(r+1)^{\text{th}}$ term of $(1+x)^{\frac{1}{2}}$.
23. The $(r+1)^{\text{th}}$ term of $(1+x)^{\frac{11}{3}}$.
24. The 14^{th} term of $(2^{10}-2^7x)^{\frac{13}{2}}$.
25. The 7^{th} term of $(3^8+6^4x)^{\frac{11}{4}}$.

183. If we expand $(1-x)^{-2}$ by the Binomial Theorem, we obtain

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots;$$

but, by referring to Art. 60, we see that this result is only true when x is less than 1. This leads us to enquire whether we are always justified in assuming the truth of the statement

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \ldots\ldots,$$

and, if not, under what conditions the expansion of $(1+x)^n$ may be used as its true equivalent.

Suppose, for instance, that $n=-1$; then we have

$$(1-x)^{-1}=1+x+x^2+x^3+x^4+\ldots\ldots\ldots\ldots (1);$$

in this equation put $x=2$; we then obtain

$$(-1)^{-1}=1+2+2^2+2^3+2^4+\ldots\ldots.$$

This contradictory result is sufficient to shew that we cannot take

$$1+nx+\frac{n(n-1)}{1.2}x^2+\ldots\ldots$$

as the true arithmetical equivalent of $(1+x)^n$ in all cases.

Now from the formula for the sum of a geometrical progression, we know that the sum of the first r terms of the series (1) $=\dfrac{1-x^r}{1-x}$

$$=\frac{1}{1-x}-\frac{x^r}{1-x};$$

and, when x is numerically less than 1, by taking r sufficiently large we can make $\dfrac{x^r}{1-x}$ as small as we please; that is, by taking a sufficient number of terms the sum can be made to differ as little as we please from $\dfrac{1}{1-x}$. But when x is numerically greater than 1, the value of $\dfrac{x^r}{1-x}$ increases with r, and therefore no such approximation to the value of $\dfrac{1}{1-x}$ is obtained by taking any number of terms of the series

$$1+x+x^2+x^3+\ldots\ldots.$$

It will be seen in the chapter on Convergency and Divergency of Series that the expansion by the Binomial Theorem of $(1+x)^n$ in ascending powers of x is always arithmetically intelligible when x is less than 1.

But if x is greater than 1, then since the general term of the series

$$1+nx+\frac{n(n-1)}{1.2}x^2+\ldots\ldots,$$

contains x^r, it can be made greater than any finite quantity by taking r sufficiently large; in which case there is no limit to the value of the above series; and therefore the expansion of $(1+x)^n$ as an infinite series in ascending powers of x has no meaning arithmetically intelligible when x is greater than 1.

184. We may remark that we can always expand $(x+y)^n$ by the Binomial Theorem; for we may write the expression in either of the two following forms:

$$x^n\left(1+\frac{y}{x}\right)^n, \qquad y^n\left(1+\frac{x}{y}\right)^n;$$

and we obtain the expansion from the first or second of these according as x is greater or less than y.

185. *To find in its simplest form the general term in the expansion of* $(1-x)^{-n}$.

The $(r+1)^{\text{th}}$ term

$$=\frac{(-n)(-n-1)(-n-2)\dots(-n-r+1)}{\lfloor r}(-x)^r$$

$$=(-1)^r\frac{n(n+1)(n+2)\dots(n+r-1)}{\lfloor r}(-1)^r x^r$$

$$=(-1)^{2r}\frac{n(n+1)(n+2)\dots(n+r-1)}{\lfloor r}x^r$$

$$=\frac{n(n+1)(n+2)\dots(n+r-1)}{\lfloor r}x^r.$$

From this it appears that every term in the expansion of $(1-x)^{-n}$ is positive.

Although the general term in the expansion of any binomial may always be found as explained in Art. 182, it will be found more expeditious in practice to use the above form of the general term in all cases where the index is negative, retaining the form

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r}x^r$$

only in the case of positive indices.

Example. Find the general term in the expansion of $\dfrac{1}{\sqrt[3]{1-3x}}$.

$$\frac{1}{\sqrt[3]{1-3x}} = (1-3x)^{-\frac{1}{3}}.$$

The $(r+1)^{\text{th}}$ term

$$= \frac{\frac{1}{3}\left(\frac{1}{3}+1\right)\left(\frac{1}{3}+2\right)\ldots\ldots\left(\frac{1}{3}+r-1\right)}{\lfloor r} (3x)^r$$

$$= \frac{1 \cdot 4 \cdot 7 \ldots\ldots (3r-2)}{3^r \lfloor r} 3^r x^r$$

$$= \frac{1 \cdot 4 \cdot 7 \ldots\ldots (3r-2)}{\lfloor r} x^r.$$

If the given expression had been $(1+3x)^{-\frac{1}{3}}$ we should have used the same formula for the general term, replacing $3x$ by $-3x$.

19!

186. The following expansions should be remembered:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots + x^r + \ldots\ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots + (r+1)\,x^r + \ldots\ldots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots + \frac{(r+1)(r+2)}{1 \cdot 2} x^r + \ldots\ldots$$

187. The general investigation of the greatest term in the expansion of $(1+x)^n$, when n is unrestricted in value, will be found in Art. 189; but the student will have no difficulty in applying to any numerical example the method explained in Art. 172.

Example. Find the greatest term in the expansion of $(1+x)^{-n}$ when $x=\frac{2}{3}$, and $n=20$.

We have
$$T_{r+1} = \frac{n+r-1}{r} \cdot x \times T_r, \textit{ numerically,}$$
$$= \frac{19+r}{r} \times \frac{2}{3} \times T_r;$$
$$\therefore\ T_{r+1} > T_r,$$
so long as
$$\frac{2(19+r)}{3r} > 1;$$
that is,
$$38 > r.$$

Hence for all values of r up to 37, we have $T_{r+1} > T_r$; but if $r=38$, then $T_{r+1} = T_r$, and these are the greatest terms. Thus the 38^{th} and 39^{th} terms are equal numerically and greater than any other term

188. Some useful applications of the Binomial Theorem are explained in the following examples.

Example 1. Find the first three terms in the expansion of

$$(1+3x)^{\frac{1}{2}}(1-2x)^{-\frac{1}{3}}.$$

Expanding the two binomials as far as the term containing x^2, we have

$$\left(1+\frac{3}{2}x-\frac{9}{8}x^2-\ldots\right)\left(1+\frac{2}{3}x+\frac{8}{9}x^2+\ldots\right)$$

$$=1+x\left(\frac{3}{2}+\frac{2}{3}\right)+x^2\left(\frac{8}{9}+\frac{3}{2}\cdot\frac{2}{3}-\frac{9}{8}\right)\ldots\ldots$$

$$=1+\frac{13}{6}x+\frac{55}{72}x^2.$$

If in this Example $x=\cdot002$, so that $x^2=\cdot000004$, we see that the third term is a decimal fraction beginning with 5 ciphers. If therefore we were required to find the numerical value of the given expression correct to 5 places of decimals it would be sufficient to substitute $\cdot002$ for x in $1+\frac{13}{6}x$, neglecting the term involving x^2.

Example 2. When x is so small that its square and higher powers may be neglected, find the value of

$$\frac{\left(1+\frac{2}{3}x\right)^{-5}+\sqrt{4+2x}}{\sqrt{(4+x)^3}}.$$

Since x^2 and the higher powers may be neglected, it will be sufficient to retain the first two terms in the expansion of each binomial. Therefore

$$\text{the expression}\quad=\frac{\left(1+\frac{2}{3}x\right)^{-5}+2\left(1+\frac{x}{2}\right)^{\frac{1}{2}}}{8\left(1+\frac{x}{4}\right)^{\frac{3}{2}}}$$

$$=\frac{\left(1-\frac{10}{3}x\right)+2\left(1+\frac{1}{4}x\right)}{8\left(1+\frac{3}{8}x\right)}$$

$$=\frac{1}{8}\left(3-\frac{17}{6}x\right)\left(1+\frac{3}{8}x\right)^{-1}$$

$$=\frac{1}{8}\left(3-\frac{17}{6}x\right)\left(1-\frac{3}{8}x\right)$$

$$=\frac{1}{8}\left(3-\frac{95}{24}x\right),$$

the term involving x^2 being neglected.

Example 3. Find the value of $\frac{1}{\sqrt{47}}$ to four places of decimals.

$$\frac{1}{\sqrt{47}}=(47)^{-\frac{1}{2}}=(7^2-2)^{-\frac{1}{2}}=\frac{1}{7}\left(1-\frac{2}{7^2}\right)^{-\frac{1}{2}}$$

$$=\frac{1}{7}\left(1+\frac{1}{7^2}+\frac{3}{2}\cdot\frac{1}{7^4}+\frac{5}{2}\cdot\frac{1}{7^6}+\dots\right)$$

$$=\frac{1}{7}+\frac{1}{7^3}+\frac{3}{2}\cdot\frac{1}{7^5}+\frac{5}{2}\cdot\frac{1}{7^7}+\dots$$

To obtain the values of the several terms we proceed as follows:

$$\begin{array}{rll}
7\,)\,1 & & \\
7\,)\,\cdot142857 & \dots\dots\dots\dots & =\frac{1}{7}, \\
7\,)\,\cdot020408 & & \\
7\,)\,\cdot002915 & \dots\dots\dots\dots & =\frac{1}{7^3}, \\
7\,)\,\cdot000416 & & \\
\cdot000059 & \dots\dots\dots\dots & =\frac{1}{7^5};
\end{array}$$

and we can see that the term $\frac{5}{2}\cdot\frac{1}{7^7}$ is a decimal fraction beginning with 5 ciphers.

$$\therefore\ \frac{1}{\sqrt{47}}=\cdot142857+\cdot002915+\cdot000088$$

$$=\cdot14586,$$

and this result is correct to at least four places of decimals.

Example 4. Find the cube root of 126 to 5 places of decimals.

$$(126)^{\frac{1}{3}}=(5^3+1)^{\frac{1}{3}}$$

$$=5\left(1+\frac{1}{5^3}\right)^{\frac{1}{3}}$$

$$=5\left(1+\frac{1}{3}\cdot\frac{1}{5^3}-\frac{1}{9}\cdot\frac{1}{5^6}+\frac{5}{81}\cdot\frac{1}{5^9}-\dots\right)$$

$$=5+\frac{1}{3}\cdot\frac{1}{5^2}-\frac{1}{9}\cdot\frac{1}{5^5}+\frac{1}{81}\cdot\frac{1}{5^7}-\dots$$

$$=5+\frac{1}{3}\cdot\frac{2^2}{10^2}-\frac{1}{9}\cdot\frac{2^5}{10^5}+\frac{1}{81}\cdot\frac{2^7}{10^7}-\dots$$

$$=5+\frac{\cdot04}{3}-\frac{\cdot00032}{9}+\frac{\cdot0000128}{81}-\dots$$

$$=5+\cdot013333\dots-\cdot000035\dots+\dots$$

$$=5\cdot01329,\text{ to five places of decimals.}$$

EXAMPLES. XIV. b.

Find the $(r+1)^{\text{th}}$ term in each of the following expansions:

1. $(1+x)^{-\frac{1}{2}}$.
2. $(1-x)^{-5}$.
3. $(1+3x)^{\frac{1}{3}}$.
4. $(1+x)^{-\frac{2}{3}}$.
5. $(1+x^2)^{-3}$.
6. $(1-2x)^{-\frac{3}{2}}$.
7. $(a+bx)^{-1}$.
8. $(2-x)^{-2}$.
9. $\sqrt[3]{(a^3-x^3)^2}$.
10. $\dfrac{1}{\sqrt{1+2x}}$.
11. $\dfrac{1}{\sqrt[3]{(1-3x)^2}}$.
12. $\dfrac{1}{\sqrt[n]{a^n-nx}}$.

Find the greatest term in each of the following expansions:

13. $(1+x)^{-7}$ when $x=\dfrac{4}{15}$.
14. $(1+x)^{\frac{21}{2}}$ when $x=\dfrac{2}{3}$.
15. $(1-7x)^{-\frac{11}{4}}$ when $x=\dfrac{1}{8}$.
16. $(2x+5y)^{12}$ when $x=8$ and $y=3$.
17. $(5-4x)^{-7}$ when $x=\dfrac{1}{2}$.
18. $(3x^2+4y^3)^{-n}$ when $x=9$, $y=2$, $n=15$.

Find to five places of decimals the value of

19. $\sqrt{98}$.
20. $\sqrt[3]{998}$.
21. $\sqrt[3]{1003}$.
22. $\sqrt[4]{2400}$.
23. $\dfrac{1}{\sqrt[3]{128}}$.
24. $(1\frac{1}{250})^{\frac{1}{3}}$.
25. $(630)^{-\frac{3}{4}}$.
26. $\sqrt[5]{3128}$.

If x be so small that its square and higher powers may be neglected, find the value of

27. $(1-7x)^{\frac{1}{3}}(1+2x)^{-\frac{3}{4}}$.
28. $\sqrt{4-x}\,.\left(3-\dfrac{x}{2}\right)^{-1}$.
29. $\dfrac{(8+3x)^{\frac{2}{3}}}{(2+3x)\sqrt{4-5x}}$.
30. $\dfrac{\left(1+\frac{2}{3}x\right)^{-5}\times(4+3x)^{\frac{1}{2}}}{(4+x)^{\frac{3}{2}}}$.

31. $\dfrac{\sqrt[4]{1-\frac{3}{5}x}+\left(1+\frac{5}{6}x\right)^{-6}}{\sqrt[3]{1+2x}+\sqrt[5]{1-\frac{x}{2}}}$.   32. $\dfrac{\sqrt[3]{8+3x}-\sqrt[5]{1-x}}{(1+5x)^{\frac{3}{5}}+\left(4+\frac{x}{2}\right)^{\frac{1}{2}}}$.

33. Prove that the coefficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\dfrac{|\underline{2r}}{(|\underline{r})^2}$.

34. Prove that $(1+x)^n = 2^n\left\{1 - n\dfrac{1-x}{1+x} + \dfrac{n(n+1)}{1 \cdot 2}\left(\dfrac{1-x}{1+x}\right)^2 \dots\dots\right\}$.

35. Find the first three terms in the expansion of

$$\frac{1}{(1+x)^2\sqrt{1+4x}}.$$

36. Find the first three terms in the expansion of

$$\frac{(1+x)^{\frac{3}{4}}+\sqrt{1+5x}}{(1-x)^2}.$$

37. Shew that the n^{th} coefficient in the expansion of $(1-x)^{-n}$ is double of the $(n-1)^{\text{th}}$.

189. *To find the numerically greatest term in the expansion of* $(1+x)^n$, *for any rational value of* n.

Since we are only concerned with the *numerical* value of the greatest term, we shall consider x throughout as positive.

Case I. Let n be a positive integer.

The $(r+1)^{\text{th}}$ term is obtained by multiplying the r^{th} term by $\dfrac{n-r+1}{r} \cdot x$; that is, by $\left(\dfrac{n+1}{r}-1\right)x$; and therefore the terms continue to increase so long as

$$\left(\frac{n+1}{r}-1\right)x > 1;$$

that is,
$$\frac{(n+1)x}{r} > 1 + x,$$

or
$$\frac{(n+1)x}{1+x} > r.$$

If $\dfrac{(n+1)x}{1+x}$ be an integer, denote it by p; then if $r=p$, the multiplying factor is 1, and the $(p+1)^{\text{th}}$ term is equal to the p^{th}, and these are greater than any other term.

If $\dfrac{(n+1)x}{1+x}$ be not an integer, denote its integral part by q; then the greatest value of r is q, and the $(q+1)^{\text{th}}$ term is the greatest.

Case II. Let n be a positive fraction.

As before, the $(r+1)^{\text{th}}$ term is obtained by multiplying the r^{th} term by $\left(\dfrac{n+1}{r}-1\right)x$.

(1) If x be greater than unity, by increasing r the above multiplier can be made as near as we please to $-x$; so that after a certain term each term is nearly x times the preceding term numerically, and thus the terms increase continually, and there is no greatest term.

(2) If x be less than unity we see that the multiplying factor continues positive, and decreases until $r>n+1$, and from this point it becomes negative but always remains less than 1 numerically; therefore there will be a greatest term.

As before, the multiplying factor will be greater than 1

so long as $$\frac{(n+1)x}{1+x}>r.$$

If $\dfrac{(n+1)x}{1+x}$ be an integer, denote it by p; then, as in Case I., the $(p+1)^{\text{th}}$ term is equal to the p^{th}, and these are greater than any other term.

If $\dfrac{(n+1)x}{1+x}$ be not an integer, let q be its integral part; then the $(q+1)^{\text{th}}$ term is the greatest.

Case III. Let n be negative.

Let $n=-m$, so that m is positive; then the numerical value of the multiplying factor is $\dfrac{m+r-1}{r}\cdot x$; that is

$$\left(\frac{m-1}{r}+1\right)x.$$

(1) If x be greater than unity we may shew, as in Case II., that there is no greatest term.

(2) If x be less than unity, the multiplying factor will be greater than 1, so long as

$$\left(\frac{m-1}{r}+1\right)x > 1;$$

that is,
$$\frac{(m-1)x}{r} > 1-x,$$

or
$$\frac{(m-1)x}{1-x} > r.$$

If $\frac{(m-1)x}{1-x}$ be a positive integer, denote it by p; then the $(p+1)^{\text{th}}$ term is equal to the p^{th} term, and these are greater than any other term.

If $\frac{(m-1)x}{1-x}$ be positive but not an integer, let q be its integral part; then the $(q+1)^{\text{th}}$ term is the greatest.

If $\frac{(m-1)x}{1-x}$ be negative, then m is less than unity; and by writing the multiplying factor in the form $\left(1-\frac{1-m}{r}\right)x$, we see that it is always less than 1: hence each term is less than the preceding, and consequently the first term is the greatest.

190. *To find the number of homogeneous products of* r *dimensions that can be formed out of the* n *letters* a, b, c, ... *and their powers.*

By division, or by the Binomial Theorem, we have

$$\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \ldots\ldots,$$

$$\frac{1}{1-bx} = 1 + bx + b^2x^2 + b^3x^3 + \ldots\ldots,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \ldots\ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence, by multiplication,

$$\frac{1}{1-ax} \cdot \frac{1}{1-bx} \cdot \frac{1}{1-cx} \cdot \ldots\ldots$$

$$= (1 + ax + a^2x^2 + \ldots)(1 + bx + b^2x^2 + \ldots)(1 + cx + c^2x^2 + \ldots)\ldots$$

$$= 1 + x(a + b + c + \ldots) + x^2(a^2 + ab + ac + b^2 + bc + c^2 + \ldots) + \ldots$$

$$= 1 + S_1x + S_2x^2 + S_3x^3 + \ldots\ldots \text{ suppose;}$$

where S_1, S_2, S_3, ... are the *sums* of the homogeneous products of *one*, *two*, *three*, ... dimensions that can be formed of a, b, c, ... and their powers.

To obtain the *number* of these products, put a, b, c, ... each equal to 1; each term in S_1, S_2, S_3, ... now becomes 1, and the values of S_1, S_2, S_3, ... so obtained give the *number* of the homogeneous products of *one*, *two*, *three*, ... dimensions.

Also $$\frac{1}{1-ax} \cdot \frac{1}{1-bx} \cdot \frac{1}{1-cx} \ldots\ldots$$

becomes $$\frac{1}{(1-x)^n} \text{ or } (1-x)^{-n}.$$

Hence S_r = coefficient of x^r in the expansion of $(1-x)^{-n}$

$$= \frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{\underline{|r}}$$

$$= \frac{\underline{|n+r-1}}{\underline{|r}\,\underline{|n-1}}.$$

191. *To find the number of terms in the expansion of any multinomial when the index is a positive integer.*

In the expansion of

$$(a_1 + a_2 + a_3 + \ldots\ldots + a_r)^n,$$

every term is of n dimensions; therefore the number of terms is the same as the number of homogeneous products of n dimensions that can be formed out of the r quantities a_1, a_2, ... a_r, and their powers; and therefore by the preceding article is equal to

$$\frac{\underline{|r+n-1}}{\underline{|n}\,\underline{|r-1}}.$$

192. From the result of Art. 190 we may deduce a theorem relating to the number of combinations of n things.

Consider n letters $a, b, c, d, \ldots\ldots$; then if we were to write down all the homogeneous products of r dimensions which can be formed of these letters and their powers, every such product would represent one of the combinations, r at a time, of the n letters, when any one of the letters might occur once, twice, thrice, ... up to r times.

Therefore the number of combinations of n things r at a time when repetitions are allowed is equal to the number of homogeneous products of r dimensions which can be formed out of n letters, and therefore equal to $\dfrac{|\underline{n+r-1}}{|\underline{r}\;|\underline{n-1}}$, or ${}^{n+r-1}C_r$.

That is, the number of combinations of n things r at a time when repetitions are allowed is equal to the number of combinations of $n+r-1$ things r at a time when repetitions are excluded.

193. We shall conclude this chapter with a few miscellaneous examples.

Example 1. Find the coefficient of x^r in the expansion of $\dfrac{(1-2x)^2}{(1+x)^3}$. $(1+x)^{-3}$

The expression $= (1-4x+4x^2)(1+p_1x+p_2x^2+\ldots+p_rx^r+\ldots)$ suppose.

The coefficient of x^r will be obtained by multiplying p_r, p_{r-1}, p_{r-2} by 1, -4, 4 respectively, and adding the results; hence

$$\text{the required coefficient} = p_r - 4p_{r-1} + 4p_{r-2}.$$

But $$p_r = (-1)^r\frac{(r+1)(r+2)}{2}. \qquad \text{[Ex. 3, Art. 182.]}$$

Hence the required coefficient

$$= (-1)^r\frac{(r+1)(r+2)}{2} - 4(-1)^{r-1}\frac{r(r+1)}{2} + 4(-1)^{r-2}\frac{(r-1)r}{2}$$

$$= \frac{(-1)^r}{2}[(r+1)(r+2) + 4r(r+1) + 4r(r-1)]$$

$$= \frac{(-1)^r}{2}(9r^2+3r+2).$$

Example 2. Find the value of the series

$$2+\frac{5}{\underline{|2}\,.\,3}+\frac{5\,.\,7}{\underline{|3}\,.\,3^2}+\frac{5\,.\,7\,.\,9}{\underline{|4}\,.\,3^3}+\ldots$$

The expression $=2+\dfrac{3\,.\,5}{\underline{|2}}\cdot\dfrac{1}{3^2}+\dfrac{3\,.\,5\,.\,7}{\underline{|3}}\cdot\dfrac{1}{3^3}+\dfrac{3\,.\,5\,.\,7\,.\,9}{\underline{|4}}\cdot\dfrac{1}{3^4}+\ldots$

$$=2+\frac{\frac{3}{2}\cdot\frac{5}{2}}{\underline{|2}}\cdot\frac{2^2}{3^2}+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{\underline{|3}}\cdot\frac{2^3}{3^3}+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{\underline{|4}}\cdot\frac{2^4}{3^4}+\ldots$$

$$=1+\frac{\frac{3}{2}}{1}\cdot\frac{2}{3}+\frac{\frac{3}{2}\cdot\frac{5}{2}}{\underline{|2}}\cdot\left(\frac{2}{3}\right)^2+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{\underline{|3}}\cdot\left(\frac{2}{3}\right)^3+\ldots$$

$$=\left(1-\frac{2}{3}\right)^{-\frac{3}{2}}=\left(\frac{1}{3}\right)^{-\frac{3}{2}}$$

$$=3^{\frac{3}{2}}=3\sqrt{3}.$$

Example 3. If n is any positive integer, shew that the integral part of $(3+\sqrt{7})^n$ is an odd number.

Suppose I to denote the integral and f the fractional part of $(3+\sqrt{7})^n$. Then $\quad I+f=3^n+C_1\,3^{n-1}\sqrt{7}+C_2\,3^{n-2}\,.\,7+C_3\,3^{n-3}(\sqrt{7})^3+\ldots\ldots\ldots\ldots(1).$

Now $3-\sqrt{7}$ is positive and less than 1, therefore $(3-\sqrt{7})^n$ is a proper fraction; denote it by f';

$$\therefore\ f'=3^n-C_1\,3^{n-1}\sqrt{7}+C_2\,3^{n-2}\,.\,7-C_3\,3^{n-3}(\sqrt{7})^3+\ldots\ldots\ldots\ldots(2).$$

Add together (1) and (2); the irrational terms disappear, and we have

$$I+f+f'=2\,(3^n+C_2\,3^{n-2}\,.\,7+\ldots)$$
$$=\text{an even integer.}$$

But since f and f' are proper fractions their sum must be 1;

$$\therefore\ I=\text{an odd integer.}$$

EXAMPLES. XIV. c.

Find the coefficient of

1. x^{100} in the expansion of $\dfrac{3-5x}{(1-x)^2}$.

2. a^{12} in the expansion of $\dfrac{4+2a-a^2}{(1+a)^3}$.

3. x^n in the expansion of $\dfrac{3x^2-2}{x+x^2}$.

4. Find the coefficient of x^n in the expansion of $\dfrac{2+x+x^2}{(1+x)^3}$.

5. Prove that

$$1-\frac{1}{2}\cdot\frac{1}{2}+\frac{1.3}{2.4}\cdot\frac{1}{2^2}-\frac{1.3.5}{2.4.6}\cdot\frac{1}{2^3}+\frac{1.3.5.7}{2.4.6.8}\cdot\frac{1}{2^4}-\dots=\sqrt{\frac{2}{3}}.$$

6. Prove that

$$\sqrt{8}=1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\dots\dots$$

7. Prove that

$$1+\frac{2n}{3}+\frac{2n(2n+2)}{3.6}+\frac{2n(2n+2)(2n+4)}{3.6.9}+\dots\dots$$
$$=2^n\left\{1+\frac{n}{3}+\frac{n(n+1)}{3.6}+\frac{n(n+1)(n+2)}{3.6.9}+\dots\dots\right\}.$$

8. Prove that

$$7^n\left\{1+\frac{n}{7}+\frac{n(n-1)}{7.14}+\frac{n(n-1)(n-2)}{7.14.21}+\dots\dots\right\}$$
$$=4^n\left\{1+\frac{n}{2}+\frac{n(n+1)}{2.4}+\frac{n(n+1)(n+2)}{2.4.6}+\dots\dots\right\}.$$

9. Prove that approximately, when x is very small,

$$\frac{3\left(x+\frac{4}{9}\right)^{\frac{1}{2}}\left(1-\frac{3}{4}x^2\right)^{\frac{1}{3}}}{2\left(1+\frac{9}{16}x\right)^2}=1-\frac{307}{256}x^2.$$

10. Shew that the integral part of $(5+2\sqrt{6})^n$ is odd, if n be a positive integer.

11. Shew that the integral part of $(8+3\sqrt{7})^n$ is odd, if n be a positive integer.

12. Find the coefficient of x^n in the expansion of

$$(1-2x+3x^2-4x^3+\dots\dots)^{-n}.$$

13. Shew that the middle term of $\left(x+\dfrac{1}{x}\right)^{4n}$ is equal to the coefficient of x^n in the expansion of $(1-4x)^{-\left(n+\frac{1}{2}\right)}$.

14. Prove that the expansion of $(1-x^3)^n$ may be put into the form

$$(1-x)^{3n}+3nx(1-x)^{3n-2}+\frac{3n(3n-3)}{1.2}x^2(1-x)^{3n-4}+\dots\dots$$

15. Prove that the coefficient of x^n in the expansion $\dfrac{1}{1+x+x^2}$ is $1, 0, -1$ according as n is of the form $3m$, $3m-1$, or $3m+1$.

16. In the expansion of $(a+b+c)^8$ find (1) the number of terms, (2) the sum of the coefficients of the terms.

17. Prove that if n be an even integer,

$$\frac{1}{1!\,(n-1)!}+\frac{1}{3!\,(n-3)!}+\frac{1}{5!\,(n-5)!}+\ldots\ldots+\frac{1}{(n-1)!\,1!}=\frac{2^{n-1}}{n!}.$$

18. If $c_0, c_1, c_2, \ldots\ldots c_n$ are the coefficients in the expansion of $(1+x)^n$, when n is a positive integer, prove that

$$(1)\quad c_0-c_1+c_2-c_3+\ldots\ldots+(-1)^r c_r=(-1)^r\frac{(n-1)!}{r!\,(n-r-1)!}.$$

$$(2)\quad c_0-2c_1+3c_2-4c_3+\ldots\ldots+(-1)^n(n+1)c_n=0.$$

$$(3)\quad c_0^2-c_1^2+c_2^2-c_3^2+\ldots\ldots+(-1)^n c_n^2=0, \text{ or } (-1)^{\frac{n}{2}}c_{\frac{n}{2}},$$

according as n is odd or even.

19. If s_n denote the sum of the first n natural numbers, prove that

$$(1)\quad (1-x)^{-3}=s_1+s_2x+s_3x^2+\ldots\ldots+s_nx^{n-1}+\ldots$$

$$(2)\quad 2(s_1s_{2n}+s_2s_{2n-1}+\ldots\ldots+s_ns_{n+1})=\frac{(2n+4)!}{5!\,(2n-1)!}.$$

20. If $q_n=\dfrac{1\,.\,3\,.\,5\,.\,7\ldots\ldots(2n-1)}{2\,.\,4\,.\,6\,.\,8\ldots\ldots2n}$, prove that

$$(1)\quad q_{2n+1}+q_1q_{2n}+q_2q_{2n-1}+\ldots\ldots+q_{n-1}q_{n+2}+q_nq_{n+1}=\frac{1}{2}.$$

$$(2)\quad 2\{q_{2n}-q_1q_{2n-1}+q_2q_{2n-2}+\ldots\ldots+(-1)^{n-1}q_{n-1}q_{n+1}\}$$
$$=q_n+(-1)^{n-1}q_n^2.$$

21. Find the sum of the products, two at a time, of the coefficients in the expansion of $(1+x)^n$, when n is a positive integer.

22. If $(7+4\sqrt{3})^n=p+\beta$, where n and p are positive integers, and β a proper fraction, shew that $(1-\beta)(p+\beta)=1$.

23. If $c_0, c_1, c_2, \ldots\ldots c_n$ are the coefficients in the expansion of $(1+x)^n$, where n is a positive integer, shew that

$$c_1-\frac{c_2}{2}+\frac{c_3}{3}-\ldots\ldots+\frac{(-1)^{n-1}c_n}{n}=1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}.$$

CHAPTER XV.

Multinomial Theorem.

194. We have already seen in Art. 175, how we may apply the Binomial Theorem to obtain the expansion of a multinomial expression. In the present chapter our object is not so much to obtain the complete expansion of a multinomial as to find the coefficient of any assigned term.

Example. Find the coefficient of $a^4b^2c^3d^5$ in the expansion of

$$(a+b+c+d)^{14}.$$

The expansion is the product of 14 factors each equal to $a+b+c+d$, and every term in the expansion is of 14 dimensions, being a product formed by taking one letter out of each of these factors. Thus to form the term $a^4b^2c^3d^5$, we take a out of any *four* of the fourteen factors, b out of any *two* of the remaining ten, c out of any *three* of the remaining eight. But the number of ways in which this can be done is clearly equal to the number of ways of arranging 14 letters when four of them must be a, two b, three c, and five d; that is, equal to

$$\frac{\lfloor 14}{\lfloor 4 \lfloor 2 \lfloor 3 \lfloor 5}. \qquad \text{[Art. 151.]}$$

This is therefore the number of times in which the term $a^4b^2c^3d^5$ appears in the final product, and consequently the coefficient required is 2522520.

195. *To find the coefficient of any assigned term in the expansion* of $(a+b+c+d+\ldots)^p$, *where* p *is a positive integer.*

The expansion is the product of p factors each equal to $a+b+c+d+\ldots$, and every term in the expansion is formed by taking one letter out of each of these p factors; and therefore the number of ways in which any term $a^\alpha b^\beta c^\gamma d^\delta \ldots$ will appear in the final product is equal to the number of ways of arranging p letters when α of them must be a, β must be b, γ must be c; and so on. That is,

$$\text{the coefficient of} \quad a^\alpha b^\beta c^\gamma d^\delta \ldots \text{ is } \frac{\lfloor p}{\lfloor \alpha \lfloor \beta \lfloor \gamma \lfloor \delta \ldots},$$

$$\text{where} \qquad \alpha+\beta+\gamma+\delta+\ldots=p.$$

COR. In the expansion of

$$(a + bx + cx^2 + dx^3 + \dots)^p,$$

the term involving $a^\alpha b^\beta c^\gamma d^\delta \dots$ is

$$\frac{\underline{|p}}{\underline{|\alpha}\, \underline{|\beta}\, \underline{|\gamma}\, \underline{|\delta} \dots} a^\alpha (bx)^\beta (cx^2)^\gamma (dx^3)^\delta \dots,$$

or

$$\frac{\underline{|p}}{\underline{|\alpha}\, \underline{|\beta}\, \underline{|\gamma}\, \underline{|\delta} \dots} a^\alpha b^\beta c^\gamma d^\delta \dots x^{\beta + 2\gamma + 3\delta + \dots},$$

where $\alpha + \beta + \gamma + \delta + \dots = p$.

This may be called *the general term* of the expansion.

Example. Find the coefficient of x^5 in the expansion of $(a + bx + cx^2)^9$.

The general term of the expansion is

$$\frac{\underline{|9}}{\underline{|\alpha}\, \underline{|\beta}\, \underline{|\gamma}} a^\alpha b^\beta c^\gamma x^{\beta + 2\gamma} \quad \dots\dots\dots\dots (1),$$

where $\alpha + \beta + \gamma = 9$.

We have to obtain by trial all the positive integral values of β and γ which satisfy the equation $\beta + 2\gamma = 5$; the values of α can then be found from the equation $$\alpha + \beta + \gamma = 9.$$

Putting $\gamma = 2$, we have $\beta = 1$, and $\alpha = 6$;
putting $\gamma = 1$, we have $\beta = 3$, and $\alpha = 5$;
putting $\gamma = 0$, we have $\beta = 5$, and $\alpha = 4$.

The required coefficient will be the sum of the corresponding values of the expression (1).

Therefore the coefficient required

$$= \frac{\underline{|9}}{\underline{|6}\, \underline{|2}} a^6bc^2 + \frac{\underline{|9}}{\underline{|5}\, \underline{|3}} a^5b^3c + \frac{\underline{|9}}{\underline{|4}\, \underline{|5}} a^4b^5$$

$$= 252a^6bc^2 + 504a^5b^3c + 126a^4b^5.$$

196. *To find the general term in the expansion of*

$$(a + bx + cx^2 + dx^3 + \dots)^n,$$

where n *is any rational quantity.*

By the Binomial Theorem, the general term is

$$\frac{n(n-1)(n-2)\dots(n-p+1)}{\underline{|p}} a^{n-p} (bx + cx^2 + dx^3 + \dots)^p,$$

where p is a positive integer.

And, by Art. 195, the general term of the expansion of

$$(bx + cx^2 + dx^3 + \ldots)^p$$

is
$$\frac{|\underline{p}}{|\underline{\beta}\,|\underline{\gamma}\,|\underline{\delta}\ldots}\, b^\beta c^\gamma d^\delta \ldots x^{\beta+2\gamma+3\delta+\ldots},$$

where β, γ, δ ... are positive integers whose sum is p.

Hence the general term in the expansion of the given expression is

$$\frac{n(n-1)(n-2)\ldots(n-p+1)}{|\underline{\beta}\,|\underline{\gamma}\,|\underline{\delta}\ldots}\, a^{n-p} b^\beta c^\gamma d^\delta \ldots x^{\beta+2\gamma+3\delta+\ldots},$$

where
$$\beta+\gamma+\delta+\ldots = p.$$

197. Since $(a + bx + cx^2 + dx^3 + \ldots)^n$ may be written in the form

$$a^n\left(1 + \frac{b}{a}x + \frac{c}{a}x^2 + \frac{d}{a}x^3 + \ldots\right)^n,$$

it will be sufficient to consider the case in which the first term of the multinomial is unity.

Thus the general term of

$$(1 + bx + cx^2 + dx^3 + \ldots)^n$$

is
$$\frac{n(n-1)(n-2)\ldots(n-p+1)}{|\underline{\beta}\,|\underline{\gamma}\,|\underline{\delta}}\, b^\beta c^\gamma d^\delta \ldots x^{\beta+2\gamma+3\delta+\ldots},$$

where
$$\beta+\gamma+\delta+\ldots = p.$$

Example. Find the coefficient of x^3 in the expansion of

$$(1-3x-2x^2+6x^3)^{\frac{2}{3}}.$$

The general term is

$$\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)\ldots\left(\frac{2}{3}-p+1\right)}{|\underline{\beta}\,|\underline{\gamma}\,|\underline{\delta}}(-3)^\beta(-2)^\gamma(6)^\delta x^{\beta+2\gamma+3\delta} \quad\ldots\ldots (1).$$

We have to obtain by trial all the positive integral values of β, γ, δ which satisfy the equation $\beta+2\gamma+3\delta=3$; and then p is found from the equation $p=\beta+\gamma+\delta$. The required coefficient will be the sum of the corresponding values of the expression (1).

In finding β, γ, δ, ... it will be best to commence by giving to δ successive integral values beginning with the greatest admissible. In the present case the values are found to be

$$\delta=1,\quad \gamma=0,\quad \beta=0,\quad p=1;$$
$$\delta=0,\quad \gamma=1,\quad \beta=1,\quad p=2;$$
$$\delta=0,\quad \gamma=0,\quad \beta=3,\quad p=3.$$

Substituting these values in (1) the required coefficient

$$=\left(\frac{2}{3}\right)(6)+\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)(-3)(-2)+\frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{\underline{\lvert 3}}(-3)^3$$
$$=4-\frac{4}{3}-\frac{4}{3}=\frac{4}{3}.$$

198. Sometimes it is more expeditious to use the Binomial Theorem.

Example. Find the coefficient of x^4 in the expansion of $(1-2x+3x^2)^{-3}$.

The required coefficient is found by picking out the coefficient of x^4 from the first few terms of the expansion of $(1-\overline{2x-3x^2})^{-3}$ by the Binomial Theorem; that is, from

$$1+3(2x-3x^2)+6(2x-3x^2)^2+10(2x-3x^2)^3+15(2x-3x^2)^4;$$

we stop at this term for all the other terms involve powers of x higher than x^4.

The required coefficient $=6\,.\,9+10\,.\,3(2)^2(-3)+15(2)^4$
$$=-66.$$

EXAMPLES. XV.

Find the coefficient of

1. $a^2b^3c^4d$ in the expansion of $(a-b-c+d)^{10}$.
2. a^2b^5d in the expansion of $(a+b-c-d)^8$.
3. a^3b^3c in the expansion of $(2a+b+3c)^7$.
4. $x^2y^3z^4$ in the expansion of $(ax-by+cz)^9$.
5. x^3 in the expansion of $(1+3x-2x^2)^3$.
6. x^4 in the expansion of $(1+2x+3x^2)^{10}$.
7. x^6 in the expansion of $(1+2x-x^2)^5$.
8. x^8 in the expansion of $(1-2x+3x^2-4x^3)^4$.

Find the coefficient of

9. x^{23} in the expansion of $(1-2x+3x^2-x^4-x^5)^5$.

10. x^5 in the expansion of $(1-2x+3x^2)^{-\frac{1}{2}}$.

11. x^3 in the expansion of $(1-2x+3x^2-4x^3)^{\frac{1}{2}}$.

12. x^8 in the expansion of $\left(1-\frac{x^2}{3}+\frac{x^4}{9}\right)^{-2}$.

13. x^4 in the expansion of $(2-4x+3x^2)^{-2}$.

14. x^6 in the expansion of $(1+4x^2+10x^4+20x^6)^{-\frac{3}{4}}$.

15. x^{12} in the expansion of $(3-15x^3+18x^6)^{-1}$.

16. Expand $(1-2x-2x^2)^{\frac{1}{4}}$ as far as x^2.

17. Expand $(1+3x^2-6x^3)^{-\frac{2}{3}}$ as far as x^5.

18. Expand $(8-9x^3+18x^4)^{\frac{4}{3}}$ as far as x^8.

19. If $(1+x+x^2+\ldots\ldots+x^p)^n=a_0+a_1x+a_2x^2+\ldots\ldots a_{np}x^{np}$, prove that

(1) $a_0+a_1+a_2+\ldots\ldots+a_{np}=(p+1)^n$.

(2) $a_1+2a_2+3a_3+\ldots\ldots+np \, . \, a_{np}=\frac{1}{2}np(p+1)^n$.

20. If $a_0, a_1, a_2, a_3 \ldots$ are the coefficients in order of the expansion of $(1+x+x^2)^n$, prove that

$$a_0^2-a_1^2+a_2^2-a_3^2+\ldots\ldots+(-1)^{n-1}a_{n-1}^2=\frac{1}{2}a_n\{1-(-1)^n a_n\}.$$

21. If the expansion of $(1+x+x^2)^n$

be $$a_0+a_1x+a_2x^2+\ \ldots\ +a_rx^r+\ \ldots\ +a_{2n}x^{2n},$$

shew that

$$a_0+a_3+a_6+\ \ldots\ =a_1+a_4+a_7+\ \ldots\ =a_2+a_5+a_8+\ \ldots\ =3^{n-1}.$$

CHAPTER XVI.

Logarithms.

199. Definition. The **logarithm** of any number to a given **base** is the index of the power to which the base must be raised in order to equal the given number. Thus if $a^x = N$, x is called the logarithm of N to the base a.

Examples. (1) Since $3^4 = 81$, the logarithm of 81 to base 3 is 4.

(2) Since $10^1 = 10$, $10^2 = 100$, $10^3 = 1000, \ldots\ldots$

the natural numbers 1, 2, 3.... are respectively the logarithms of 10, 100, 1000,...... to base 10.

200. The logarithm of N to base a is usually written $\log_a N$, so that the same meaning is expressed by the two equations

$$a^x = N; \quad x = \log_a N.$$

From these equations we deduce

$$N = a^{\log_a N},$$

an identity which is sometimes useful.

Example. Find the logarithm of $32\sqrt[5]{4}$ to base $2\sqrt{2}$.

Let x be the required logarithm; then,

by definition, $$(2\sqrt{2})^x = 32\sqrt[5]{4};$$

$$\therefore\ (2 \cdot 2^{\frac{1}{2}})^x = 2^5 \cdot 2^{\frac{2}{5}};$$

$$\therefore\ 2^{\frac{3}{2}x} = 2^{5+\frac{2}{5}};$$

hence, by equating the indices, $$\frac{3}{2}x = \frac{27}{5};$$

$$\therefore\ x = \frac{18}{5} = 3 \cdot 6.$$

201. When it is understood that a particular system of logarithms is in use, the suffix denoting the base is omitted. Thus in arithmetical calculations in which 10 is the base, we usually write $\log 2$, $\log 3$,...... instead of $\log_{10} 2$, $\log_{10} 3$,......

Any number might be taken as the base of logarithms, and corresponding to any such base a system of logarithms of all numbers could be found. But before discussing the logarithmic systems commonly used, we shall prove some general propositions which are true for all logarithms independently of any particular base.

202. *The logarithm of* 1 *is* 0.

For $a^0 = 1$ for all values of a; therefore $\log 1 = 0$, whatever the base may be.

203. *The logarithm of the base itself is* 1.

For $a^1 = a$; therefore $\log_a a = 1$.

204. *To find the logarithm of a product.*

Let MN be the product; let a be the base of the system, and suppose

$$x = \log_a M, \qquad y = \log_a N;$$

so that $$a^x = M, \qquad a^y = N.$$

Thus the product $$MN = a^x \times a^y$$
$$= a^{x+y};$$

whence, by definition, $$\log_a MN = x + y$$
$$= \log_a M + \log_a N.$$

Similarly, $\log_a MNP = \log_a M + \log_a N + \log_a P$;

and so on for any number of factors.

Example. $$\log 42 = \log (2 \times 3 \times 7)$$
$$= \log 2 + \log 3 + \log 7.$$

205. *To find the logarithm of a fraction.*

Let $\dfrac{M}{N}$ be the fraction, and suppose

$$x = \log_a M, \qquad y = \log_a N;$$

so that $$a^x = M, \qquad a^y = N.$$

Thus the fraction $$\frac{M}{N} = \frac{a^x}{a^y}$$

$$= a^{x-y};$$

whence, by definition, $$\log_a \frac{M}{N} = x - y$$

$$= \log_a M - \log_a N.$$

Example. $$\log (4\tfrac{2}{7}) = \log \frac{30}{7}$$

$$= \log 30 - \log 7$$

$$= \log (2 \times 3 \times 5) - \log 7$$

$$= \log 2 + \log 3 + \log 5 - \log 7.$$

206. *To find the logarithm of a number raised to any power, integral or fractional.*

Let $\log_a(M^p)$ be required, and suppose

$$x = \log_a M, \text{ so that } a^x = M;$$

then $$M^p = (a^x)^p$$

$$= a^{px};$$

whence, by definition, $$\log_a(M^p) = px;$$

that is, $$\log_a(M^p) = p \log_a M.$$

Similarly, $$\log_a(M^{\frac{1}{r}}) = \frac{1}{r} \log_a M.$$

207. It follows from the results we have proved that

(1) the logarithm of a product is equal to the sum of the logarithms of its factors;

(2) the logarithm of a fraction is equal to the logarithm of the numerator diminished by the logarithm of the denominator;

(3) the logarithm of the p^{th} power of a number is p times the logarithm of the number;

(4) the logarithm of the r^{th} root of a number is equal to $\frac{1}{r}$th of the logarithm of the number.

Also we see that by the use of logarithms the operations of multiplication and division may be replaced by those of addition and subtraction; and the operations of involution and evolution by those of multiplication and division.

Example 1. Express the logarithm of $\frac{\sqrt{a^3}}{c^5b^2}$ in terms of $\log a$, $\log b$ and $\log c$.

$$\log\frac{\sqrt{a^3}}{c^5b^2}=\log\frac{a^{\frac{3}{2}}}{c^5b^2}$$

$$=\log a^{\frac{3}{2}}-\log(c^5b^2)$$

$$=\frac{3}{2}\log a-(\log c^5+\log b^2)$$

$$=\frac{3}{2}\log a-5\log c-2\log b.$$

Example 2. Find x from the equation $a^x . c^{-2x}=b^{3x+1}$.

Taking logarithms of both sides, we have

$$x\log a-2x\log c=(3x+1)\log b;$$

$$\therefore\ x(\log a-2\log c-3\log b)=\log b;$$

$$\therefore\ x=\frac{\log b}{\log a-2\log c-3\log b}.$$

EXAMPLES. XVI. a.

Find the logarithms of

1. 16 to base $\sqrt{2}$, and 1728 to base $2\sqrt{3}$.

2. 125 to base $5\sqrt{5}$, and $\cdot 25$ to base 4.

3. $\frac{1}{256}$ to base $2\sqrt{2}$, and $\cdot\dot{3}$ to base 9.

4. $\cdot 0625$ to base 2, and 1000 to base $\cdot 01$.

5. $\cdot 0001$ to base $\cdot 001$, and $\cdot\dot{1}$ to base $9\sqrt{3}$.

6. $\sqrt[4]{a^{\frac{8}{5}}}$, $\frac{1}{a^{-\frac{1}{2}}}$, $\sqrt[3]{a^{-\frac{15}{2}}}$ to base a.

7. Find the value of

$$\log_8 128,\quad \log_6\frac{1}{216},\quad \log_{27}\frac{1}{81},\quad \log_{343}49.$$

Express the following seven logarithms in terms of $\log a$, $\log b$, and $\log c$.

8. $\log(\sqrt{a^2b^3})^6$. 9. $\log(\sqrt[3]{a^2}\times\sqrt[2]{b^3})$. 10. $\log(\sqrt[9]{a^{-4}b^3})$.

11. $\log\left(\sqrt{a^{-2}b} \times \sqrt[3]{ab^{-3}}\right)$.

12. $\log\left(\sqrt[3]{a^{-1}\sqrt{b^3}} \div \sqrt{b^3\sqrt{a}}\right)$.

13. $\log \dfrac{\sqrt[3]{ab^{-1}c^{-2}}}{(a^{-1}b^{-2}c^{-4})^{\frac{1}{6}}}$.

14. $\log\left\{\left(\dfrac{bc^{-2}}{b^{-4}c^3}\right)^{-3} \div \left(\dfrac{b^{-1}c}{b^2c^{-3}}\right)^5\right\}$.

15. Shew that $\log \dfrac{\sqrt[4]{5}\,.\,\sqrt[10]{2}}{\sqrt[3]{18}\,.\,\sqrt{2}} = \dfrac{1}{4}\log 5 - \dfrac{2}{5}\log 2 - \dfrac{2}{3}\log 3$.

16. Simplify $\quad \log \sqrt[4]{729\sqrt[3]{9^{-1}\,.\,27^{-\frac{4}{3}}}}$.

17. Prove that $\log\dfrac{75}{16} - 2\log\dfrac{5}{9} + \log\dfrac{32}{243} = \log 2$.

Solve the following equations:

18. $a^x = cb^x$.

19. $a^{2x}\,.\,b^{3x} = c^5$.

20. $\dfrac{a^{x+1}}{b^{x-1}} = c^{2x}$.

21. $\left.\begin{array}{l} a^{2x}\,.\,b^{3y} = m^5 \\ a^{3x}\,.\,b^{2y} = m^{10} \end{array}\right\}$.

22. If $\log(x^2y^3) = a$, and $\log\dfrac{x}{y} = b$, find $\log x$ and $\log y$.

23. If $a^{3-x}\,.\,b^{5x} = a^{x+5}\,.\,b^{3x}$, shew that $x\log\left(\dfrac{b}{a}\right) = \log a$.

24. Solve the equation

$$(a^4 - 2a^2b^2 + b^4)^{x-1} = (a-b)^{2x}(a+b)^{-2}.$$

Common Logarithms.

208. Logarithms to the base 10 are called **Common Logarithms**; this system was first introduced, in 1615, by Briggs, a contemporary of Napier the inventor of logarithms.

From the equation $10^x = N$, it is evident that common logarithms will not in general be integral, and that they will not always be positive.

For instance $\qquad 3154 > 10^3$ and $< 10^4$;

$$\therefore \log 3154 = 3 + \text{a fraction}.$$

Again, $\cdot 06 > 10^{-2}$ and $< 10^{-1}$;

$$\therefore \log \cdot 06 = -2 + \text{a fraction.}$$

209. Definition. The integral part of a logarithm is called the **characteristic**, and the decimal part is called the **mantissa.**

The characteristic of the logarithm of any number to the base 10 can be written down by inspection, as we shall now shew.

210. *To determine the characteristic of the logarithm of any number greater than unity.*

Since
$$10^1 = 10,$$
$$10^2 = 100,$$
$$10^3 = 1000,$$
$$\cdots\cdots\cdots\cdots$$

it follows that a number with two digits in its integral part lies between 10^1 and 10^2; a number with three digits in its integral part lies between 10^2 and 10^3; and so on. Hence a number with n digits in its integral part lies between 10^{n-1} and 10^n.

Let N be a number whose integral part contains n digits; then

$$N = 10^{(n-1) + \text{a fraction}};$$

$$\therefore \log N = (n-1) + \text{a fraction.}$$

Hence the characteristic is $n - 1$; that is, *the characteristic of the logarithm of a number greater than unity is less by one than the number of digits in its integral part, and is positive.*

211. *To determine the characteristic of the logarithm of a decimal fraction.*

Since
$$10^0 = 1,$$
$$10^{-1} = \frac{1}{10} = \cdot 1,$$
$$10^{-2} = \frac{1}{100} = \cdot 01,$$
$$10^{-3} = \frac{1}{1000} = \cdot 001,$$
$$\cdots\cdots\cdots\cdots$$

it follows that a decimal with one cipher immediately after the decimal point, such as ·0324, being greater than ·01 and less than ·1, lies between 10^{-2} and 10^{-1}; a number with two ciphers after the decimal point lies between 10^{-3} and 10^{-2}; and so on. Hence a decimal fraction with n ciphers immediately after the decimal point lies between $10^{-(n+1)}$ and 10^{-n}.

Let D be a decimal beginning with n ciphers; then

$$D = 10^{-(n+1) + \text{a fraction}};$$

$$\therefore \log D = -(n+1) + \text{a fraction}.$$

Hence the characteristic is $-(n+1)$; that is, *the characteristic of the logarithm of a decimal fraction is greater by unity than the number of ciphers immediately after the decimal point, and is negative.*

212. The logarithms to base 10 of all integers from 1 to 200000 have been found and tabulated; in most Tables they are given to seven places of decimals. This is the system in practical use, and it has two great advantages:

(1) From the results already proved it is evident that the characteristics can be written down by inspection, so that only the mantissæ have to be registered in the Tables.

(2) The mantissæ are the same for the logarithms of all numbers which have the same significant digits; so that it is sufficient to tabulate the mantissæ of the logarithms of *integers*.

This proposition we proceed to prove.

213. Let N be any number, then since multiplying or dividing by a power of 10 merely alters the position of the decimal point without changing the sequence of figures, it follows that $N \times 10^p$, and $N \div 10^q$, where p and q are any integers, are numbers whose significant digits are the same as those of N.

Now $\quad \log(N \times 10^p) = \log N + p \log 10$

$$= \log N + p \quad \text{..........................(1)}.$$

Again, $\quad \log(N \div 10^q) = \log N - q \log 10$

$$= \log N - q \quad \text{..........................(2)}.$$

In (1) an integer is added to $\log N$, and in (2) an integer is subtracted from $\log N$; that is, the mantissa or decimal portion of the logarithm remains unaltered.

In this and the three preceding articles the mantissæ have been supposed positive. In order to secure the advantages of Briggs' system, we arrange our work so as *always to keep the mantissa positive*, so that when the mantissa of any logarithm has been taken from the Tables the characteristic is prefixed with its appropriate sign according to the rules already given.

214. In the case of a negative logarithm the minus sign is written *over the characteristic*, and not before it, to indicate that the characteristic alone is negative, and not the whole expression. Thus $\bar{4}\cdot 30103$, the logarithm of $\cdot 0002$, is equivalent to $-4 + \cdot 30103$, and must be distinguished from $-4\cdot 30103$, an expression in which both the integer and the decimal are negative. In working with negative logarithms an arithmetical artifice will sometimes be necessary in order to make the mantissa positive. For instance, a result such as $-3\cdot 69897$, in which the whole expression is negative, may be transformed by subtracting 1 from the characteristic and adding 1 to the mantissa. Thus

$$-3\cdot 69897 = -4 + (1 - \cdot 69897) = \bar{4}\cdot 30103.$$

Other cases will be noticed in the Examples.

Example 1. Required the logarithm of $\cdot 0002432$.

In the Tables we find that 3859636 is the mantissa of log 2432 (the decimal point as well as the characteristic being omitted); and, by Art. 211, the characteristic of the logarithm of the given number is -4;

$$\therefore \log \cdot 0002432 = \bar{4}\cdot 3859636.$$

Example 2. Find the value of $\sqrt[5]{\cdot 00000165}$, given

$$\log 165 = 2\cdot 2174839,\ \log 697424 = 5\cdot 8434968.$$

Let x denote the value required; then

$$\log x = \log (\cdot 00000165)^{\frac{1}{5}} = \frac{1}{5} \log (\cdot 00000165)$$

$$= \frac{1}{5} (\bar{6}\cdot 2174839);$$

the *mantissa* of log $\cdot 00000165$ being the same as that of log 165, and the *characteristic* being prefixed by the rule.

Now $$\frac{1}{5} (\bar{6}\cdot 2174839) = \frac{1}{5} (\overline{10} + 4\cdot 2174839)$$

$$= \bar{2}\cdot 8434968$$

and $\cdot 8434968$ is the mantissa of $\log 697424$; hence x is a number consisting of these same digits but with one cipher after the decimal point. [Art. 211.]

Thus $$x = \cdot 0697424.$$

215. The method of calculating logarithms will be explained in the next chapter, and it will there be seen that they are first found to another base, and then transformed into common logarithms to base 10.

It will therefore be necessary to investigate a method for transforming a system of logarithms having a given base to a new system with a different base.

216. Suppose that the logarithms of all numbers to base a are known and tabulated, it is required to find the logarithms to base b.

Let N be any number whose logarithm to base b is required.

Let $y = \log_b N$, so that $b^y = N$;

$$\therefore \quad \log_a (b^y) = \log_a N;$$

that is,
$$y \log_a b = \log_a N;$$

$$\therefore \quad y = \frac{1}{\log_a b} \times \log_a N,$$

or
$$\log_b N = \frac{1}{\log_a b} \times \log_a N \text{ ....}(1).$$

Now since N and b are given, $\log_a N$ and $\log_a b$ are known from the Tables, and thus $\log_b N$ may be found.

Hence it appears that to transform logarithms from base a to base b we have only to multiply them all by $\frac{1}{\log_a b}$; this is a constant quantity and is given by the Tables; it is known as the *modulus*.

217. In equation (1) of the preceding article put a for N; thus

$$\log_b a = \frac{1}{\log_a b} \times \log_a a = \frac{1}{\log_a b};$$

$$\therefore \quad \log_b a \times \log_a b = 1.$$

This result may also be proved directly as follows:

Let $$x = \log_a b, \text{ so that } a^x = b;$$

then by taking logarithms to base b, we have

$$x \log_b a = \log_b b$$
$$= 1;$$
$$\therefore \quad \log_a b \times \log_b a = 1.$$

218. The following examples will illustrate the utility of logarithms in facilitating arithmetical calculation; but for information as to the use of Logarithmic Tables the reader is referred to works on Trigonometry.

Example 1. Given $\log 3 = \cdot 4771213$, find $\log \{(2\cdot 7)^3 \times (\cdot 81)^{\frac{4}{5}} \div (90)^{\frac{5}{4}}\}$.

The required value $$= 3 \log \frac{27}{10} + \frac{4}{5} \log \frac{81}{100} - \frac{5}{4} \log 90$$
$$= 3(\log 3^3 - 1) + \frac{4}{5}(\log 3^4 - 2) - \frac{5}{4}(\log 3^2 + 1)$$
$$= \left(9 + \frac{16}{5} - \frac{5}{2}\right) \log 3 - \left(3 + \frac{8}{5} + \frac{5}{4}\right)$$
$$= \frac{97}{10} \log 3 - 5\tfrac{17}{20}$$
$$= 4\cdot 6280766 - 5\cdot 85$$
$$= \bar{2}\cdot 7780766.$$

The student should notice that the logarithm of 5 and its powers can always be obtained from $\log 2$; thus

$$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - \log 2.$$

Example 2. Find the number of digits in 875^{16}, given

$$\log 2 = \cdot 3010300, \ \log 7 = \cdot 8450980.$$

$$\log (875^{16}) = 16 \log (7 \times 125)$$
$$= 16 (\log 7 + 3 \log 5)$$
$$= 16 (\log 7 + 3 - 3 \log 2)$$
$$= 16 \times 2\cdot 9420080$$
$$= 47\cdot 072128;$$

hence the number of digits is 48. [Art. 210.]

Example 3. Given log 2 and log 3, find to two places of decimals the value of x from the equation

$$6^{3-4x} \cdot 4^{x+5} = 8.$$

Taking logarithms of both sides, we have

$$(3-4x)\log 6 + (x+5)\log 4 = \log 8;$$

$$\therefore\ (3-4x)(\log 2 + \log 3) + (x+5)\,2\log 2 = 3\log 2;$$

$$\therefore\ x(-4\log 2 - 4\log 3 + 2\log 2) = 3\log 2 - 3\log 2 - 3\log 3 - 10\log 2;$$

$$\therefore\ x = \frac{10\log 2 + 3\log 3}{2\log 2 + 4\log 3}$$

$$= \frac{4{\cdot}4416639}{2{\cdot}5105452}$$

$$= 1{\cdot}77\ldots$$

EXAMPLES. XVI. b.

1. Find, by inspection, the characteristics of the logarithms of 21735, 23·8, 350, ·035, ·2, ·87, ·875.

2. The mantissa of log 7623 is ·8821259; write down the logarithms of 7·623, 762·3, ·007623, 762300, ·000007623.

3. How many digits are there in the integral part of the numbers whose logarithms are respectively

4·30103, 1·4771213, 3·69897, ·56515?

4. Give the position of the first significant figure in the numbers whose logarithms are

$\bar{2}{\cdot}7781513$, $\cdot 6910815$, $\bar{5}{\cdot}4871384$.

Given $\log 2 = {\cdot}3010300$, $\log 3 = {\cdot}4771213$, $\log 7 = {\cdot}8450980$, find the value of

5. $\log 64$. **6.** $\log 84$. **7.** $\log {\cdot}128$.

8. $\log {\cdot}0125$. **9.** $\log 14{\cdot}4$. **10.** $\log 4\frac{2}{3}$.

11. $\log \sqrt[3]{12}$. **12.** $\log \sqrt{\dfrac{35}{27}}$. **13.** $\log \sqrt[4]{{\cdot}0105}$.

14. Find the seventh root of ·00324, having given that

$$\log 44092388 = 7{\cdot}6443636.$$

15. Given $\log 194{\cdot}8445 = 2{\cdot}2896883$, find the eleventh root of $(39{\cdot}2)^2$.

16. Find the product of 37·203, 3·7203, ·0037203, 372030, having given that

$$\log 37{\cdot}203 = 1{\cdot}5705780, \text{ and } \log 1915631 = 6{\cdot}2823120.$$

17. Given log 2 and log 3, find $\log \sqrt[3]{\left(\dfrac{3^2 5^4}{\sqrt{2}}\right)}$.

18. Given log 2 and log 3, find $\log\left(\sqrt[3]{48} \times 108^{\frac{1}{4}} \div \sqrt[12]{6}\right)$.

19. Calculate to six decimal places the value of

$$\sqrt[3]{\left(\frac{294 \times 125}{42 \times 32}\right)^2};$$

given log 2, log 3, log 7; also $\log 9076{\cdot}226 = 3{\cdot}9579053$.

20. Calculate to six places of decimals the value of

$$(330 \div 49)^4 \div \sqrt[3]{22 \times 70};$$

given log 2, log 3, log 7; also

$$\log 11 = 1{\cdot}0413927, \text{ and } \log 17814{\cdot}1516 = 4{\cdot}2507651.$$

21. Find the number of digits in $3^{12} \times 2^8$.

22. Shew that $\left(\dfrac{21}{20}\right)^{100}$ is greater than 100.

23. Determine how many ciphers there are between the decimal point and the first significant digit in $\left(\dfrac{1}{2}\right)^{1000}$.

Solve the following equations, having given log 2, log 3, and log 7.

24. $3^{x-2} = 5$. **25.** $5^x = 10^3$. **26.** $5^{5-3x} = 2^{x+2}$.

27. $21^x = 2^{2x+1} . 5^x$. **28.** $2^x . 6^{x-2} = 5^{2x} . 7^{1-x}$.

29. $\left.\begin{aligned} 2^{x+y} &= 6^y \\ 3^x &= 3 . 2^{y+1} \end{aligned}\right\}$. **30.** $\left.\begin{aligned} 3^{1-x-y} &= 4^{-y} \\ 2^{2x-1} &= 3^{3y-x} \end{aligned}\right\}$.

31. Given $\log_{10} 2 = {\cdot}30103$, find $\log_{25} 200$.

32. Given $\log_{10} 2 = {\cdot}30103$, $\log_{10} 7 = {\cdot}84509$, find $\log_7 \sqrt{2}$ and $\log_{\sqrt{2}} 7$.

CHAPTER XVII.

EXPONENTIAL AND LOGARITHMIC SERIES.

219. IN Chap. XVI. it was stated that the logarithms in common use were not found directly, but that logarithms are first found to another base, and then transformed to base 10.

In the present chapter we shall prove certain formulæ known as the **Exponential and Logarithmic Series**, and give a brief explanation of the way in which they are used in constructing a table of logarithms.

220. *To expand* a^x *in ascending powers of* x.

By the Binomial Theorem, *if* n *is greater than* 1,

$$\left(1+\frac{1}{n}\right)^{nx}$$

$$=1+nx\cdot\frac{1}{n}+\frac{nx(nx-1)}{\underline{|2}}\cdot\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{\underline{|3}}\cdot\frac{1}{n^3}+\dots\dots$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\underline{|2}}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\underline{|3}}+\dots\dots\dots\dots\dots\dots\dots(1).$$

By putting $x=1$, we obtain

$$\left(1+\frac{1}{n}\right)^{n}=1+1+\frac{1-\frac{1}{n}}{\underline{|2}}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}}+\dots\dots(2).$$

But $$\left(1+\frac{1}{n}\right)^{nx}=\left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{x};$$

hence the series (1) is the x^{th} power of the series (2); that is,

$$1 + x + \frac{x\left(x - \frac{1}{n}\right)}{\underline{|2}} + \frac{x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)}{\underline{|3}} + \ldots\ldots$$

$$= \left\{1 + 1 + \frac{1 - \frac{1}{n}}{\underline{|2}} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{\underline{|3}} + \ldots\right\}^x;$$

and this is true however great n may be. If therefore n be indefinitely increased we have

$$1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots\ldots = \left(1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \ldots\ldots\right)^x.$$

The series $$1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \ldots\ldots$$

is usually denoted by e; hence

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots\ldots.$$

Write cx for x, then

$$e^{cx} = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \frac{c^3x^3}{\underline{|3}} + \ldots\ldots.$$

Now let $e^c = a$, so that $c = \log_e a$; by substituting for c we obtain

$$a^x = 1 + x \log_e a + \frac{x^2 (\log_e a)^2}{\underline{|2}} + \frac{x^3 (\log_e a)^3}{\underline{|3}} + \ldots\ldots.$$

This is the *Exponential Theorem.*

COR. When n is infinite, the *limit* of $\left(1 + \frac{1}{n}\right)^n = e$.

[See Art. 266.]

Also as in the preceding investigation, it may be shewn that when n is indefinitely increased,

$$\left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots\ldots;$$

that is, when n is infinite, the limit of $\left(1+\frac{x}{n}\right)^n = e^x$.

By putting $\frac{x}{n} = -\frac{1}{m}$, we have

$$\left(1-\frac{x}{n}\right)^n = \left(1+\frac{1}{m}\right)^{-mx} = \left\{\left(1+\frac{1}{m}\right)^m\right\}^{-x}.$$

Now m is infinite when n is infinite;

thus the limit of $$\left(1-\frac{x}{n}\right)^n = e^{-x}.$$

Hence the limit of $$\left(1-\frac{1}{n}\right)^n = e^{-1}.$$

221. In the preceding article no restriction is placed upon the value of x; also since $\frac{1}{n}$ is less than unity, the expansions we have used give results arithmetically intelligible. [Art. 183.]

But there is another point in the foregoing proof which deserves notice. We have assumed that when n is infinite

the limit of $$\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\dots\left(x-\frac{r-1}{n}\right)}{\underline{|r}} \text{ is } \frac{x^r}{\underline{|r}}$$

for all values of r.

Let us denote the value of

$$\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\dots\left(x-\frac{r-1}{n}\right)}{\underline{|r}} \text{ by } u_r.$$

Then $$\frac{u_r}{u_{r-1}} = \frac{1}{r}\left(x-\frac{r-1}{n}\right) = \frac{x}{r} - \frac{1}{n} + \frac{1}{nr}.$$

Since n is infinite, we have

$$\frac{u_r}{u_{r-1}} = \frac{x}{r}; \text{ that is, } u_r = \frac{x}{r}u_{r-1}.$$

It is clear that the limit of u_2 is $\frac{x^2}{\underline{|2}}$; hence the limit of u_3 is $\frac{x^3}{\underline{|3}}$; that of u_4 is $\frac{x^4}{\underline{|4}}$; and generally that of u_r is $\frac{x^r}{\underline{|r}}$.

222. The series

$$1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\ldots\ldots,$$

which we have denoted by e, is very important as it is the base to which logarithms are first calculated. Logarithms to this base are known as the Napierian system, so named after Napier their inventor. They are also called *natural* logarithms from the fact that they are the first logarithms which naturally come into consideration in algebraical investigations.

When logarithms are used in theoretical work it is to be remembered that the base e is always understood, just as in arithmetical work the base 10 is invariably employed.

From the series the approximate value of e can be determined to any required degree of accuracy; to 10 places of decimals it is found to be 2·7182818284.

Example 1. Find the sum of the infinite series

$$1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\ldots\ldots$$

We have $$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\ldots\ldots;$$

and by putting $x=-1$ in the series for e^x,

$$e^{-1}=1-1+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}-\ldots\ldots$$

$$\therefore\ e+e^{-1}=2\left(1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\ldots\ldots\right);$$

hence the sum of the series is $\frac{1}{2}(e+e^{-1})$.

Example 2. Find the coefficient of x^r in the expansion of $\frac{1-ax-x^2}{e^x}$.

$$\frac{1-ax-x^2}{e^x}=(1-ax-x^2)\,e^{-x}$$

$$=(1-ax-x^2)\left\{1-x+\frac{x^2}{\underline{|2}}-\frac{x^3}{\underline{|3}}+\ldots+\frac{(-1)^r x^r}{\underline{|r}}+\ldots\ldots\right\}.$$

The coefficient required $= \dfrac{(-1)^r}{\underline{|r}} - \dfrac{(-1)^{r-1}a}{\underline{|r-1}} - \dfrac{(-1)^{r-2}}{\underline{|r-2}}$

$$= \frac{(-1)^r}{\underline{|r}} \{1 + ar - r(r-1)\}.$$

223. *To expand* $\log_e(1+x)$ *in ascending powers of* x.

From Art. 220,

$$a^y = 1 + y \log_e a + \frac{y^2 (\log_e a)^2}{\underline{|2}} + \frac{y^3 (\log_e a)^3}{\underline{|3}} + \ldots\ldots$$

In this series write $1 + x$ for a; thus

$$(1+x)^y$$
$$= 1 + y \log_e (1+x) + \frac{y^2}{\underline{|2}} \{\log_e (1+x)\}^2 + \frac{y^3}{\underline{|3}} \{\log_e (1+x)\}^3 + \ldots (1).$$

Also by the Binomial Theorem, when $x < 1$ we have

$$(1+x)^y = 1 + yx + \frac{y(y-1)}{\underline{|2}} x^2 + \frac{y(y-1)(y-2)}{\underline{|3}} x^3 + \ldots\ldots (2).$$

Now in (2) the coefficient of y is

$$x + \frac{(-1)}{1 \cdot 2} x^2 + \frac{(-1)(-2)}{1 \cdot 2 \cdot 3} x^3 + \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \ldots\ldots ;$$

that is, $$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots .$$

Equate this to the coefficient of y in (1); thus we have

$$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots .$$

This is known as the *Logarithmic Series.*

Example. If $x < 1$, expand $\{\log_e (1+x)\}^2$ in ascending powers of x.

By equating the coefficients of y^2 in the series (1) and (2), we see that the required expansion is double the coefficient of y^2 in

$$\frac{y(y-1)}{1 \cdot 2} x^2 + \frac{y(y-1)(y-2)}{1 \cdot 2 \cdot 3} x^3 + \frac{y(y-1)(y-2)(y-3)}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \ldots\ldots ;$$

that is, double the coefficient of y in

$$\frac{y-1}{1 \cdot 2} x^2 + \frac{(y-1)(y-2)}{1 \cdot 2 \cdot 3} x^3 + \frac{(y-1)(y-2)(y-3)}{1 \cdot 2 \cdot 3 \cdot 4} x + \ldots\ldots .$$

Thus $\{\log_e (1+x)\}^2 = 2 \left\{ \dfrac{1}{2} x^2 - \dfrac{1}{3}\left(1 + \dfrac{1}{2}\right) x^3 + \dfrac{1}{4}\left(1 + \dfrac{1}{2} + \dfrac{1}{3}\right) x^4 - \ldots\ldots \right\}.$

224. Except when x is very small the series for $\log_e(1+x)$ is of little use for numerical calculations. We can, however, deduce from it other series by the aid of which Tables of Logarithms may be constructed.

By writing $\frac{1}{n}$ for x we obtain $\log_e \frac{n+1}{n}$; hence

$$\log_e(n+1) - \log_e n = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\dots\dots\dots(1).$$

By writing $-\frac{1}{n}$ for x we obtain $\log_e \frac{n-1}{n}$; hence, by changing signs on both sides of the equation,

$$\log_e n - \log_e(n-1) = \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \dots\dots\dots\dots(2).$$

From (1) and (2) by addition,

$$\log_e(n+1) - \log_e(n-1) = 2\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \dots\right) \dots\dots(3).$$

From this formula by putting $n=3$ we obtain $\log_e 4 - \log_e 2$, that is $\log_e 2$; and by effecting the calculation we find that the value of $\log_e 2 = \cdot 69314718\dots$; whence $\log_e 8$ is known.

Again by putting $n=9$ we obtain $\log_e 10 - \log_e 8$; whence we find $\log_e 10 = 2\cdot 30258509\dots$.

To convert Napierian logarithms into logarithms to base 10 we multiply by $\frac{1}{\log_e 10}$, which is the *modulus* [Art. 216] of the common system, and its value is $\frac{1}{2\cdot 30258509\dots}$, or $\cdot 43429448\dots$; we shall denote this modulus by μ.

In the *Proceedings of the Royal Society of London*, Vol. XXVII. page 88, Professor J. C. Adams has given the values of e, μ, $\log_e 2$, $\log_e 3$, $\log_e 5$ to more than 260 places of decimals.

225. If we multiply the above series throughout by μ, we obtain formulæ adapted to the calculation of *common logarithms*. Thus from (1), $\mu\log_e(n+1) - \mu\log_e n = \frac{\mu}{n} - \frac{\mu}{2n^2} + \frac{\mu}{3n^3} - \dots$;

that is,

$$\log_{10}(n+1) - \log_{10} n = \frac{\mu}{n} - \frac{\mu}{2n^2} + \frac{\mu}{3n^3} - \dots\dots\dots\dots (1).$$

Similarly from (2),

$$\log_{10} n - \log_{10}(n-1) = \frac{\mu}{n} + \frac{\mu}{2n^2} + \frac{\mu}{3n^3} + \dots\dots\dots\dots (2).$$

From either of the above results we see that if the logarithm of one of two consecutive numbers be known, the logarithm of the other may be found, and thus a table of logarithms can be constructed.

It should be remarked that the above formulæ are only needed to calculate the logarithms of *prime* numbers, for the logarithm of a *composite* number may be obtained by adding together the logarithms of its component factors.

In order to calculate the logarithm of any one of the smaller prime numbers, we do not usually substitute the number in either of the formulæ (1) or (2), but we endeavour to find some value of n by which division may be easily performed, and such that either $n+1$ or $n-1$ contains the given number as a factor. We then find $\log(n+1)$ or $\log(n-1)$ and deduce the logarithm of the given number.

Example. Calculate log 2 and log 3, given $\mu = \cdot 43429448$.

By putting $n = 10$ in (2), we have the value of $\log 10 - \log 9$; thus

$$1 - 2\log 3 = \cdot 043429448 + \cdot 002171472 + \cdot 000144765 + \cdot 000010857$$
$$+ \cdot 000000868 + \cdot 000000072 + \cdot 000000006;$$

$$1 - 2\log 3 = \cdot 045757488,$$

$$\log 3 = \cdot 477121256.$$

Putting $n = 80$ in (1), we obtain $\log 81 - \log 80$; thus

$$4\log 3 - 3\log 2 - 1 = \cdot 005428681 - \cdot 000033929 + \cdot 000000283 - \cdot 000000003;$$

$$3\log 2 = \cdot 908485024 - \cdot 005395032,$$

$$\log 2 = \cdot 301029997.$$

In the next article we shall give another series for $\log_e(n+1) - \log_e n$ which is often useful in the construction of Logarithmic Tables. For further information on the subject the reader is referred to Mr Glaisher's article on *Logarithms* in the *Encyclopædia Britannica*.

226. In Art. 223 we have proved that

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots ;$$

changing x into $-x$, we have

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots .$$

By subtraction,

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right).$$

Put $\dfrac{1+x}{1-x} = \dfrac{n+1}{n}$, so that $x = \dfrac{1}{2n+1}$; we thus obtain

$$\log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots\right\}.$$

NOTE. This series converges very rapidly, but in practice is not always so convenient as the series in Art. 224.

227. The following examples illustrate the subject of the chapter.

Example 1. If α, β are the roots of the equation $ax^2+bx+c=0$, shew that $\log(a-bx+cx^2) = \log a + (\alpha+\beta)x - \dfrac{\alpha^2+\beta^2}{2}x^2 + \dfrac{\alpha^3+\beta^3}{3}x^3 - \dots$

Since $\alpha+\beta = -\dfrac{b}{a}$, $\alpha\beta = \dfrac{c}{a}$, we have

$$a - bx + cx^2 = a\{1 + (\alpha+\beta)x + \alpha\beta x^2\}$$
$$= a(1+\alpha x)(1+\beta x).$$

$$\therefore \log(a - bx + cx^2) = \log a + \log(1+\alpha x) + \log(1+\beta x)$$

$$= \log a + \alpha x - \frac{\alpha^2 x^2}{2} + \frac{\alpha^3 x^3}{3} - \dots + \beta x - \frac{\beta^2 x^2}{2} + \frac{\beta^3 x^3}{3} - \dots$$

$$= \log a + (\alpha+\beta)x - \frac{\alpha^2+\beta^2}{2}x^2 + \frac{\alpha^3+\beta^3}{3}x^3 - \dots$$

Example 2. Prove that the coefficient of x^n in the expansion of $\log(1+x+x^2)$ is $-\dfrac{2}{n}$ or $\dfrac{1}{n}$ according as n is or is not a multiple of 3.

$$\log(1+x+x^2) = \log\frac{1-x^3}{1-x} = \log(1-x^3) - \log(1-x)$$

$$= -x^3 - \frac{x^6}{2} - \frac{x^9}{3} - \dots - \frac{x^{3r}}{r} - \dots + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^r}{r} + \dots\right).$$

If n is a multiple of 3, denote it by $3r$; then the coefficient of x^n is $-\frac{1}{r}$ from the first series, together with $\frac{1}{3r}$ from the second series; that is, the coefficient is $-\frac{3}{n}+\frac{1}{n}$, or $-\frac{2}{n}$.

If n is not a multiple of 3, x^n does not occur in the first series, therefore the required coefficient is $\frac{1}{n}$.

228. *To prove that* e *is incommensurable.*

For if not, let $e=\frac{m}{n}$, where m and n are positive integers;

then
$$\frac{m}{n}=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots+\frac{1}{\underline{|n}}+\frac{1}{\underline{|n+1}}+\ldots.$$

multiply both sides by $\underline{|n}$;

$$\therefore\ m\,\underline{|n-1}=\text{integer}+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots.$$

But
$$\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots$$

is a proper fraction, for it is greater than $\frac{1}{n+1}$ and less than the geometrical progression

$$\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\ldots;$$

that is, less than $\frac{1}{n}$; hence an integer is equal to an integer plus a fraction, which is absurd; therefore e is incommensurable.

EXAMPLES. XVII.

1. Find the value of

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots.$$

2. Find the value of

$$\frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}-\ldots.$$

3. Shew that

$$\log_e(n+a) - \log_e(n-a) = 2\left(\frac{a}{n} + \frac{a^3}{3n^3} + \frac{a^5}{5n^5} + \dots\right).$$

4. If $$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots,$$

shew that $$x = y + \frac{y^2}{\lfloor 2} + \frac{y^3}{\lfloor 3} + \dots.$$

5. Shew that

$$\frac{a-b}{a} + \frac{1}{2}\left(\frac{a-b}{a}\right)^2 + \frac{1}{3}\left(\frac{a-b}{a}\right)^3 + \dots = \log_e a - \log_e b.$$

6. Find the Napierian logarithm of $\frac{1001}{999}$ correct to sixteen places of decimals.

7. Prove that $$e^{-1} = 2\left(\frac{1}{\lfloor 3} + \frac{2}{\lfloor 5} + \frac{3}{\lfloor 7} + \dots\right).$$

8. Prove that

$$\log_e(1+x)^{1+x}(1-x)^{1-x} = 2\left(\frac{x^2}{1 \cdot 2} + \frac{x^4}{3 \cdot 4} + \frac{x^6}{5 \cdot 6} + \dots\right).$$

9. Find the value of

$$x^2 - y^2 + \frac{1}{\lfloor 2}(x^4 - y^4) + \frac{1}{\lfloor 3}(x^6 - y^6) + \dots.$$

10. Find the numerical values of the common logarithms of 7, 11 and 13; given $\mu = \cdot 43429448$, $\log 2 = \cdot 30103000$.

11. Shew that if ax^2 and $\frac{a}{x^2}$ are each less than unity

$$a\left(x^2 + \frac{1}{x^2}\right) - \frac{a^2}{2}\left(x^4 + \frac{1}{x^4}\right) + \frac{a^3}{3}\left(x^6 + \frac{1}{x^6}\right) - \dots = \log_e\left(1 + ax^2 + a^2 + \frac{a}{x^2}\right).$$

12. Prove that

$$\log_e(1 + 3x + 2x^2) = 3x - \frac{5x^2}{2} + \frac{9x^3}{3} - \frac{17x^4}{4} + \dots;$$

and find the general term of the series.

13. Prove that

$$\log_e \frac{1+3x}{1-2x} = 5x - \frac{5x^2}{2} + \frac{35x^3}{3} - \frac{65x^4}{4} + \dots;$$

and find the general term of the series.

14. Expand $\frac{e^{5x} + e^x}{e^{3x}}$ in a series of ascending powers of x.

15. Express $\frac{1}{2}(e^{ix}+e^{-ix})$ in ascending powers of x, where $i=\sqrt{-1}$.

16. Shew that

$$\log_e(x+2h)=2\log_e(x+h)-\log_e x-\left\{\frac{h^2}{(x+h)^2}+\frac{h^4}{2(x+h)^4}+\frac{h^6}{3(x+h)^6}+\dots\right\}.$$

17. If α and β be the roots of $x^2-px+q=0$, shew that

$$\log_e(1+px+qx^2)=(\alpha+\beta)x-\frac{\alpha^2+\beta^2}{2}x^2+\frac{\alpha^3+\beta^3}{3}x^3-\dots.$$

18. If $x<1$, find the sum of the series

$$\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{4}x^4+\frac{4}{5}x^5+\dots.$$

19. Shew that

$$\log_e\left(1+\frac{1}{n}\right)^n=1-\frac{1}{2(n+1)}-\frac{1}{2.3(n+1)^2}-\frac{1}{3.4(n+1)^3}-\dots.$$

20. If $\log_e\frac{1}{1+x+x^2+x^3}$ be expanded in a series of ascending powers of x, shew that the coefficient of x^n is $-\frac{1}{n}$ if n be odd, or of the form $4m+2$, and $\frac{3}{n}$ if n be of the form $4m$.

21. Shew that

$$1+\frac{2^3}{\underline{|2}}+\frac{3^3}{\underline{|3}}+\frac{4^3}{\underline{|4}}+\dots=5e.$$

22. Prove that

$$2\log_e n-\log_e(n+1)-\log_e(n-1)=\frac{1}{n^2}+\frac{1}{2n^4}+\frac{1}{3n^6}+\dots.$$

23. Shew that
$$\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\dots$$
$$=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\dots.$$

24. If $\log_e\frac{9}{10}=-a$, $\log_e\frac{24}{25}=-b$, $\log_e\frac{81}{80}=c$, shew that

$$\log_e 2=7a-2b+3c,\ \log_e 3=11a-3b+5c,\ \log_e 5=16a-4b+7c;$$

and calculate $\log_e 2$, $\log_e 3$, $\log_e 5$ to 8 places of decimals.

CHAPTER XVIII.

INTEREST AND ANNUITIES.

229. In this chapter we shall explain how the solution of questions connected with Interest and Discount may be simplified by the use of algebraical formulæ.

We shall use the terms *Interest*, *Discount*, *Present Value* in their ordinary arithmetical sense; but instead of taking as the rate of interest the interest on £100 for one year, we shall find it more convenient to take the interest on £1 for one year.

230. *To find the interest and amount of a given sum in a given time at simple interest.*

Let P be the principal in pounds, r the interest of £1 for one year, n the number of years, I the interest, and M the amount.

The interest of P for one year is Pr, and therefore for n years is Pnr; that is,

$$I = Pnr \quad\quad (1).$$

Also $$M = P + I;$$

that is, $$M = P(1 + nr) \quad\quad (2).$$

From (1) and (2) we see that if of the quantities P, n, r, I, or P, n, r, M, any three be given the fourth may be found.

231. *To find the present value and discount of a given sum due in a given time, allowing simple interest.*

Let P be the given sum, V the present value, D the discount, r the interest of £1 for one year, n the number of years.

Since V is the sum which put out to interest at the present time will in n years amount to P, we have

$$P = V(1 + nr);$$

$$\therefore \quad V = \frac{P}{1 + nr}.$$

And
$$D = P - V = P - \frac{P}{1 + nr};$$

$$\therefore \quad D = \frac{Pnr}{1 + nr}.$$

NOTE. The value of D given by this equation is called the *true discount*. But in practice when a sum of money is paid before it is due, it is customary to deduct the *interest* on the debt instead of the true discount, and the money so deducted is called the *banker's discount*; so that

$$\text{Banker's Discount} = Pnr.$$

$$\text{True Discount} = \frac{Pnr}{1 + nr}.$$

Example. The difference between the true discount and the banker's discount on £1900 paid 4 months before it is due is 6*s.* 8*d.*; find the rate per cent., simple interest being allowed.

Let r denote the interest on £1 for one year; then the banker's discount is $\frac{1900r}{3}$, and the true discount is $\dfrac{\frac{1900r}{3}}{1 + \frac{1}{3}r}$.

$$\therefore \quad \frac{1900r}{3} - \frac{\frac{1900r}{3}}{1 + \frac{1}{3}r} = \frac{1}{3};$$

whence
$$1900r^2 = 3 + r;$$

$$\therefore \quad r = \frac{1 \pm \sqrt{1 + 22800}}{3800} = \frac{1 \pm 151}{3800}.$$

Rejecting the negative value, we have $r = \frac{152}{3800} = \frac{1}{25}$;

$$\therefore \text{ rate per cent.} = 100r = 4.$$

232. *To find the interest and amount of a given sum in a given time at compound interest.*

Let P denote the principal, R the amount of £1 in one year, n the number of years, I the interest, and M the amount.

The amount of P at the end of the first year is PR; and, since this is the principal for the second year, the amount at the end of the second year is $PR \times R$ or PR^2. Similarly the amount at the end of the third year is PR^3, and so on; hence the amount in n years is PR^n; that is,

$$M = PR^n;$$

$$\therefore \quad I = P(R^n - 1).$$

NOTE. If r denote the interest on £1 for one year, we have

$$R = 1 + r.$$

233. In business transactions when the time contains a fraction of a year it is usual to allow *simple* interest for the fraction of the year. Thus the amount of £1 in $\frac{1}{2}$ year is reckoned $1 + \frac{r}{2}$; and the amount of P in $4\frac{2}{3}$ years at compound interest is $PR^4\left(1 + \frac{2}{3}r\right)$. Similarly the amount of P in $n + \frac{1}{m}$ years is $PR^n\left(1 + \frac{r}{m}\right)$.

If the interest is payable more than once a year there is a distinction between the *nominal annual rate* of interest and that actually received, which may be called the *true annual rate*; thus if the interest is payable twice a year, and if r is the *nominal* annual rate of interest, the amount of £1 in half a year is $1 + \frac{r}{2}$, and therefore in the whole year the amount of £1 is $\left(1 + \frac{r}{2}\right)^2$, or $1 + r + \frac{r^2}{4}$; so that the *true* annual rate of interest is $r + \frac{r^2}{4}$.

234. If the interest is payable q times a year, and if r is the nominal annual rate, the interest on £1 for each interval is $\frac{r}{q}$, and therefore the amount of P in n years, or qn intervals, is $P\left(1 + \frac{r}{q}\right)^{qn}$.

In this case the interest is said to be "converted into principal" q times a year.

If the interest is convertible into principal every moment, then q becomes infinitely great. To find the value of the amount, put $\frac{r}{q}=\frac{1}{x}$, so that $q=rx$; thus

$$\text{the amount}=P\left(1+\frac{r}{q}\right)^{qn}=P\left(1+\frac{1}{x}\right)^{xnr}=P\left\{\left(1+\frac{1}{x}\right)^{x}\right\}^{nr}$$
$$=Pe^{nr}, \text{ [Art. 220, Cor.,]}$$

since x is infinite when q is infinite.

235. *To find the present value and discount of a given sum due in a given time, allowing compound interest.*

Let P be the given sum, V the present value, D the discount, R the amount of £1 for one year, n the number of years.

Since V is the sum which, put out to interest at the present time, will in n years amount to P, we have

$$P=VR^n;$$

$$\therefore \quad V=\frac{P}{R^n}=PR^{-n},$$

and $$D=P(1-R^{-n}).$$

Example. The present value of £672 due in a certain time is £126; if compound interest at $4\frac{1}{6}$ per cent. be allowed, find the time; having given

$$\log 2=\cdot 30103, \log 3=\cdot 47712.$$

Here $$r=\frac{4\frac{1}{6}}{100}=\frac{1}{24}, \text{ and } R=\frac{25}{24}.$$

Let n be the number of years; then

$$672=126\left(\frac{25}{24}\right)^n;$$

$$\therefore n\log\frac{25}{24}=\log\frac{672}{126},$$

or $$n\log\frac{100}{96}=\log\frac{16}{3};$$

$$\therefore n(\log 100-\log 96)=\log 16-\log 3,$$

$$n=\frac{4\log 2-\log 3}{2-5\log 2-\log 3}$$

$$n=\frac{\cdot 72700}{\cdot 01773}=41, \text{ very nearly};$$

thus the time is very nearly 41 years.

EXAMPLES. XVIII. a.

When required the following logarithms may be used.

$$\log 2 = \cdot 3010300, \quad \log 3 = \cdot 4771213,$$
$$\log 7 = \cdot 8450980, \quad \log 11 = 1 \cdot 0413927.$$

1. Find the amount of £100 in 50 years, at 5 per cent. compound interest; given $\log 114 \cdot 674 = 2 \cdot 0594650$.

2. At simple interest the interest on a certain sum of money is £90, and the discount on the same sum for the same time and at the same rate is £80; find the sum.

3. In how many years will a sum of money double itself at 5 per cent. compound interest?

4. Find, correct to a farthing, the present value of £10000 due 8 years hence at 5 per cent. compound interest; given

$$\log 67683 \cdot 94 = 4 \cdot 8304856.$$

5. In how many years will £1000 become £2500 at 10 per cent. compound interest?

6. Shew that at simple interest the discount is half the harmonic mean between the sum due and the interest on it.

7. Shew that money will increase more than a hundredfold in a century at 5 per cent. compound interest.

8. What sum of money at 6 per cent. compound interest will amount to £1000 in 12 years? Given

$$\log 106 = 2 \cdot 0253059, \quad \log 49697 = 4 \cdot 6963292.$$

9. A man borrows £600 from a money-lender, and the bill is renewed every half-year at an increase of 18 per cent.: what time will elapse before it reaches £6000? Given $\log 118 = 2 \cdot 071882$.

10. What is the amount of a farthing in 200 years at 6 per cent. compound interest? Given $\log 106 = 2 \cdot 0253059$, $\log 115 \cdot 1270 = 2 \cdot 0611800$.

Annuities.

236. An **annuity** is a fixed sum paid periodically under certain stated conditions; the payment may be made either once a year or at more frequent intervals. Unless it is otherwise stated we shall suppose the payments annual.

An **annuity certain** is an annuity payable for a fixed term of years independent of any contingency; a **life annuity** is an annuity which is payable during the lifetime of a person, or of the survivor of a number of persons.

A **deferred annuity**, or **reversion**, is an annuity which does not begin until after the lapse of a certain number of years; and when the annuity is deferred for n years, it is said to commence after n years, and the first payment is made at the end of $n+1$ years.

If the annuity is to continue for ever it is called a **perpetuity**; if it does not commence at once it is called a **deferred perpetuity.**

An annuity left unpaid for a certain number of years is said to be **forborne** for that number of years.

237. *To find the amount of an annuity left unpaid for a given number of years, allowing simple interest.*

Let A be the annuity, r the interest of £1 for one year, n the number of years, M the amount.

At the end of the first year A is due, and the amount of this sum in the remaining $n-1$ years is $A+(n-1)rA$; at the end of the second year another A is due, and the amount of this sum in the remaining $(n-2)$ years is $A+(n-2)rA$; and so on. Now M is the sum of all these amounts;

$$\therefore\ M=\{A+(n-1)rA\}+\{A+(n-2)rA\}+\ldots\ldots+(A+rA)+A,$$

the series consisting of n terms;

$$\therefore\ M=nA+(1+2+3+\ldots\ldots+\overline{n-1})rA$$

$$=nA+\frac{n(n-1)}{2}rA.$$

238. *To find the amount of an annuity left unpaid for a given number of years, allowing compound interest.*

Let A be the annuity, R the amount of £1 for one year, n the number of years, M the amount.

At the end of the first year A is due, and the amount of this sum in the remaining $n-1$ years is AR^{n-1}; at the end of the second year another A is due, and the amount of this sum in the remaining $n-2$ years is AR^{n-2}; and so on.

$$\therefore\ M=AR^{n-1}+AR^{n-2}+\ldots\ldots+AR^2+AR+A$$

$$=A(1+R+R^2+\ldots\ldots\text{ to } n \text{ terms})$$

$$=A\frac{R^n-1}{R-1}.$$

239. In finding the present value of annuities it is always customary to reckon compound interest; the results obtained when simple interest is reckoned being contradictory and untrustworthy. On this point and for further information on the subject of annuities the reader may consult the Text-books of the Institute of Actuaries, Parts I. and II., and the article *Annuities* in the *Encyclopædia Britannica.*

240. *To find the present value of an annuity to continue for a given number of years, allowing compound interest.*

Let A be the annuity, R the amount of £1 in one year, n the number of years, V the required present value.

The present value of A due in 1 year is AR^{-1};

the present value of A due in 2 years is AR^{-2};

the present value of A due in 3 years is AR^{-3};

and so on. [Art. 235.]

Now V is the sum of the present values of the different payments;

$$\therefore\ V = AR^{-1} + AR^{-2} + AR^{-3} + \dots\dots \text{to } n \text{ terms}$$

$$= AR^{-1}\frac{1-R^{-n}}{1-R^{-1}}$$

divide by R^{-1}

$$= A\,\frac{1-R^{-n}}{R-1}.$$

Note. This result may also be obtained by dividing the value of M, given in Art. 238, by R^n. [Art. 232.]

Cor. If we make n infinite we obtain for the present value of a *perpetuity*

$$V = \frac{A}{R-1} = \frac{A}{r}.$$

241. If mA is the present value of an annuity A, the annuity is said to be worth m **years' purchase.**

In the case of a perpetual annuity $mA = \dfrac{A}{r}$; hence

$$m = \frac{1}{r} = \frac{100}{\text{rate per cent.}};$$

that is, the number of years' purchase of a perpetual annuity is obtained by dividing 100 by the rate per cent.

As instances of perpetual annuities we may mention the income arising from investments in irredeemable Stocks such as many Government Securities, Corporation Stocks, and Railway Debentures. A good test of the credit of a Government is furnished by the number of years' purchase of its Stocks; thus the $2\frac{1}{2}$ p. c. Consols at 90 are worth 36 years' purchase; Egyptian 4 p. c. Stock at 96 is worth 24 years' purchase; while Austrian 5 p. c. Stock at 80 is only worth 16 years' purchase.

242. *To find the present value of a deferred annuity to commence at the end of* p *years and to continue for* n *years, allowing compound interest.*

Let A be the annuity, R the amount of £1 in one year, V the present value.

The first payment is made at the end of $(p+1)$ years. [Art. 236.]

Hence the present values of the first, second, third ... payments are respectively

$$AR^{-(p+1)},\ AR^{-(p+2)},\ AR^{-(p+3)},\ \ldots$$

$$\therefore\ V = AR^{-(p+1)} + AR^{-(p+2)} + AR^{-(p+3)} + \ldots\ldots \text{ to } n \text{ terms}$$

$$= AR^{-(p+1)}\frac{1-R^{-n}}{1-R^{-1}}$$

$$= \frac{AR^{-p}}{R-1} - \frac{AR^{-p-n}}{R-1}.$$

Cor. The present value of a *deferred perpetuity* to commence after p years is given by the formula

$$V = \frac{AR^{-p}}{R-1}.$$

243. A **freehold estate** is an estate which yields a perpetual annuity called the *rent*; and thus the value of the estate is equal to the present value of a perpetuity equal to the rent.

It follows from Art. 241 that if we know the number of years' purchase that a tenant pays in order to buy his farm, we obtain the rate per cent. at which interest is reckoned by dividing 100 by the number of years' purchase.

Example. The reversion after 6 years of a freehold estate is bought for £20000; what rent ought the purchaser to receive, reckoning compound interest at 5 per cent.? Given $\log 1{\cdot}05 = {\cdot}0211893$, $\log 1{\cdot}340096 = {\cdot}1271358$.

The rent is equal to the annual value of the perpetuity, deferred for 6 years, which may be purchased for £20000.

Let £A be the value of the annuity; then since $R = 1{\cdot}05$, we have

$$20000 = \frac{A \times (1{\cdot}05)^{-6}}{{\cdot}05};$$

$$\therefore\ A \times (1{\cdot}05)^{-6} = 1000;$$

$$\log A - 6 \log 1{\cdot}05 = 3,$$

$$\log A = 3{\cdot}1271358 = \log 1340{\cdot}096.$$

$\therefore\ A = 1340{\cdot}096$, and the rent is £1340. 1*s.* 11*d.*

244. Suppose that a tenant by paying down a certain sum has obtained a lease of an estate for $p + q$ years, and that when q years have elapsed he wishes to renew the lease for a term $p + n$ years; the sum that he must pay is called the **fine** for renewing n years of the lease.

Let A be the annual value of the estate; then since the tenant has paid for p of the $p + n$ years, the fine must be equal to the present value of a deferred annuity A, to commence after p years and to continue for n years; that is,

$$\text{the fine} = \frac{AR^{-p}}{R-1} - \frac{AR^{-p-n}}{R-1}. \qquad \text{[Art. 242.]}$$

EXAMPLES. XVIII. b.

The interest is supposed compound unless the contrary is stated.

1. The amount of an annuity of £120 which is left unpaid for 5 years is £672; find the rate per cent. allowing simple interest.

2. Find the amount of an annuity of £100 in 20 years, allowing compound interest at $4\frac{1}{2}$ per cent. Given

$$\log 1{\cdot}045 = {\cdot}0191163, \quad \log 24{\cdot}117 = 1{\cdot}3823260.$$

3. A freehold estate is bought for £2750; at what rent should it be let so that the owner may receive 4 per cent. on the purchase money?

4. A freehold estate worth £120 a year is sold for £4000; find the rate of interest.

5. How many years' purchase should be given for a freehold estate, interest being calculated at $3\frac{1}{2}$ per cent.?

$\frac{100}{25} = 4\%$

6. If a perpetual annuity is worth 25 years' purchase, find the amount of an annuity of £625 to continue for 2 years.

7. If a perpetual annuity is worth 20 years' purchase, find the annuity to continue for 3 years which can be purchased for £2522.

8. When the rate of interest is 4 per cent., find what sum must be paid now to receive a freehold estate of £400 a year 10 years hence; having given $\log 104 = 2\cdot0170333$, $\log 6\cdot75565 = \cdot8296670$.

9. Find what sum will amount to £500 in 50 years at 2 per cent., interest being payable every moment; given $e^{-1} = \cdot3678$.

10. If 25 years' purchase must be paid for an annuity to continue n years, and 30 years' purchase for an annuity to continue $2n$ years, find the rate per cent.

11. A man borrows £5000 at 4 per cent. compound interest; if the principal and interest are to be repaid by 10 equal annual instalments, find the amount of each instalment; having given

$$\log 1\cdot04 = \cdot0170333 \text{ and } \log 675565 = 5\cdot829667.$$

12. A man has a capital of £20000 for which he receives interest at 5 per cent.; if he spends £1800 every year, shew that he will be ruined before the end of the 17th year; having given

$$\log 2 = \cdot3010300, \quad \log 3 = \cdot4771213, \quad \log 7 = \cdot8450980.$$

13. The annual rent of an estate is £500; if it is let on a lease of 20 years, calculate the fine to be paid to renew the lease when 7 years have elapsed allowing interest at 6 per cent.; having given

$$\log 106 = 2\cdot0253059, \quad \log 4\cdot688385 = \cdot6710233, \quad \log 3\cdot118042 = \cdot4938820.$$

14. If a, b, c years' purchase must be paid for an annuity to continue n, $2n$, $3n$ years respectively; shew that

$$a^2 - ab + b^2 = ac.$$

15. What is the present worth of a perpetual annuity of £10 payable at the end of the first year, £20 at the end of the second, £30 at the end of the third, and so on, increasing £10 each year; interest being taken at 5 per cent. per annum?

CHAPTER XIX.

INEQUALITIES.

245. Any quantity a is said to be greater than another quantity b when $a-b$ is positive; thus 2 is greater than -3, because $2-(-3)$, or 5 is positive. Also b is said to be less than a when $b-a$ is negative; thus -5 is less than -2, because $-5-(-2)$, or -3 is negative.

In accordance with this definition, zero must be regarded as greater than any negative quantity.

In the present chapter we shall suppose (unless the contrary is directly stated) that the letters always denote real and positive quantities.

246. If $a > b$, then it is evident that

$$a+c > b+c;$$

$$a-c > b-c;$$

$$ac > bc;$$

$$\frac{a}{c} > \frac{b}{c};$$

that is, *an inequality will still hold after each side has been increased, diminished, multiplied, or divided by the same* positive *quantity.*

247. If $$a-c > b,$$
by adding c to each side,

$$a > b+c;$$

which shews that *in an inequality any term may be transposed from one side to the other if its sign be changed.*

If $a > b$, then evidently $b < a$;
that is, *if the sides of an inequality be transposed, the sign of inequality must be reversed.*

If $a > b$, then $a - b$ is positive, and $b - a$ is negative; that is, $-a - (-b)$ is negative, and therefore

$$-a < -b;$$

hence, *if the signs of all the terms of an inequality be changed, the sign of inequality must be reversed.*

Again, if $a > b$, then $-a < -b$, and therefore

$$-ac < -bc;$$

that is, *if the sides of an inequality be multiplied by the same negative quantity, the sign of inequality must be reversed.*

248. If $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$, ... $a_m > b_m$, it is clear that

$$a_1 + a_2 + a_3 + \ldots + a_m > b_1 + b_2 + b_3 + \ldots + b_m;$$

and $$a_1 a_2 a_3 \ldots a_m > b_1 b_2 b_3 \ldots b_m.$$

249. If $a > b$, and if p, q are positive integers, then $\sqrt[q]{a} > \sqrt[q]{b}$, or $a^{\frac{1}{q}} > b^{\frac{1}{q}}$; and therefore $a^{\frac{p}{q}} > b^{\frac{p}{q}}$; that is, $a^n > b^n$, where n is any positive quantity.

Further, $\dfrac{1}{a^n} < \dfrac{1}{b^n}$; that is $a^{-n} < b^{-n}$.

250. The square of every real quantity is positive, and therefore greater than zero. Thus $(a-b)^2$ is positive;

$$\therefore \quad a^2 - 2ab + b^2 > 0;$$

$$\therefore \quad a^2 + b^2 > 2ab.$$

Similarly $$\frac{x+y}{2} > \sqrt{xy};$$

that is, *the arithmetic mean of two positive quantities is greater than their geometric mean.*

The inequality becomes an equality when the quantities are equal.

251. The results of the preceding article will be found very useful, especially in the case of inequalities in which the letters are involved symmetrically.

Example 1. If a, b, c denote positive quantities, prove that

$$a^2+b^2+c^2>bc+ca+ab;$$

and $$2(a^3+b^3+c^3)>bc(b+c)+ca(c+a)+ab(a+b).$$

For $$b^2+c^2>2bc \quad\text{.......................}\quad (1);$$
$$c^2+a^2>2ca;$$
$$a^2+b^2>2ab;$$

whence by addition $$a^2+b^2+c^2>bc+ca+ab.$$

It may be noticed that this result is true for *any* real values of a, b, c.

Again, from (1) $$b^2-bc+c^2>bc \quad\text{.......................}\quad (2);$$
$$\therefore\ b^3+c^3>bc(b+c) \quad\text{.......................}\quad (3).$$

By writing down the two similar inequalities and adding, we obtain

$$2(a^3+b^3+c^3)>bc(b+c)+ca(c+a)+ab(a+b).$$

It should be observed that (3) is obtained from (2) by introducing the factor $b+c$, and that if this factor be negative the inequality (3) will no longer hold.

Example 2. If x may have any real value find which is the greater, x^3+1 or x^2+x.

$$\begin{aligned} x^3+1-(x^2+x) &= x^3-x^2-(x-1) \\ &= (x^2-1)(x-1) \\ &= (x-1)^2(x+1). \end{aligned}$$

Now $(x-1)^2$ is positive, hence

$$x^3+1 > \text{ or } < x^2+x$$

according as $x+1$ is positive or negative; that is, according as $x >$ or < -1.

If $x=-1$, the inequality becomes an equality.

252. Let a and b be two positive quantities, S their sum and P their product; then from the identity

$$4ab=(a+b)^2-(a-b)^2,$$

we have

$$4P=S^2-(a-b)^2, \text{ and } S^2=4P+(a-b)^2.$$

Hence, if S is given, P is greatest when $a=b$; and if P is given, S is least when

$$a=b;$$

that is, *if the sum of two positive quantities is given, their product is greatest when they are equal;* and *if the product of two positive quantities is given, their sum is least when they are equal.*

253. *To find the greatest value of a product the sum of whose factors is constant.*

Let there be n factors $a, b, c, \ldots k$, and suppose that their sum is constant and equal to s.

Consider the product $abc \ldots k$, and suppose that a and b are any two unequal factors. If we replace the two unequal factors a, b by the two equal factors $\frac{a+b}{2}, \frac{a+b}{2}$ the product is increased while the sum remains unaltered; hence *so long as the product contains two unequal factors it can be increased without altering the sum of the factors;* therefore the product is greatest when all the factors are equal. In this case the value of each of the n factors is $\frac{s}{n}$, and the greatest value of the product is $\left(\frac{s}{n}\right)^n$, or

$$\left(\frac{a+b+c+\ldots+k}{n}\right)^n$$

Cor. If $a, b, c, \ldots k$ are *unequal*,

$$\left(\frac{a+b+c+\ldots+k}{n}\right)^n > abc \ldots k;$$

that is,

$$\frac{a+b+c+\ldots+k}{n} > (abc \ldots k)^{\frac{1}{n}}.$$

By an extension of the meaning of the terms *arithmetic mean* and *geometric mean* this result is usually quoted as follows:

the arithmetic mean of any number of positive quantities is greater than the geometric mean.

Example. Shew that $(1^r+2^r+3^r+\ldots+n^r)^n > n^n(\underline{|n})^r$; where r is any real quantity.

Since $$\frac{1^r+2^r+3^r+\ldots+n^r}{n} > (1^r \,.\, 2^r \,.\, 3^r \ldots\ldots n^r)^{\frac{1}{n}};$$

$$\therefore \left(\frac{1^r+2^r+3^r+\ldots+n^r}{n}\right)^n > 1^r \,.\, 2^r \,.\, 3^r \ldots\ldots n^r, \text{ that is, } > (\underline{|n})^r;$$

whence we obtain the result required.

254. *To find the greatest value of* $a^m b^n c^p \dots$ *when* $a + b + c + \dots$ *is constant;* $m, n, p, \dots$ *being positive integers.*

Since $m, n, p, \dots$ are constants, the expression $a^m b^n c^p \dots$ will be greatest when $\left(\frac{a}{m}\right)^m \left(\frac{b}{n}\right)^n \left(\frac{c}{p}\right)^p \dots$ is greatest. But this last expression is the product of $m + n + p + \dots$ factors whose sum is $m\left(\frac{a}{m}\right) + n\left(\frac{b}{n}\right) + p\left(\frac{c}{p}\right) + \dots$, or $a + b + c + \dots$, and therefore constant. Hence $a^m b^n c^p \dots$ will be greatest when the factors

$$\frac{a}{m}, \ \frac{b}{n}, \ \frac{c}{p}, \dots$$

are all equal, that is, when

$$\frac{a}{m} = \frac{b}{n} = \frac{c}{p} = \dots = \frac{a + b + c + \dots}{m + n + p + \dots}.$$

Thus the greatest value is

$$m^m n^n p^p \dots \left(\frac{a + b + c + \dots}{m + n + p + \dots}\right)^{m+n+p+\dots}$$

Example. Find the greatest value of $(a+x)^3 (a-x)^4$ for any real value of x numerically less than a.

The given expression is greatest when $\left(\frac{a+x}{3}\right)^3 \left(\frac{a-x}{4}\right)^4$ is greatest; but the sum of the factors of this expression is $3\left(\frac{a+x}{3}\right) + 4\left(\frac{a-x}{4}\right)$, or $2a$; hence $(a+x)^3 (a-x)^4$ is greatest when $\frac{a+x}{3} = \frac{a-x}{4}$, or $x = -\frac{a}{7}$.

Thus the greatest value is $\frac{6^3 \cdot 8^4}{7^7} a^7$.

255. The determination of *maximum* and *minimum* values may often be more simply effected by the solution of a quadratic equation than by the foregoing methods. Instances of this have already occurred in Chap. IX.; we add a further illustration.

Example. Divide an odd integer into two integral parts whose product is a maximum.

Denote the integer by $2n+1$; the two parts by x and $2n+1-x$; and the product by y; then $(2n+1)x - x^2 = y$; whence

$$2x = (2n+1) \pm \sqrt{(2n+1)^2 - 4y};$$

but the quantity under the radical must be positive, and therefore y cannot be greater than $\frac{1}{4}(2n+1)^2$, or $n^2+n+\frac{1}{4}$; and since y is integral its greatest value must be n^2+n; in which case $x=n+1$, or n; thus the two parts are n and $n+1$.

256. Sometimes we may use the following method.

Example. Find the minimum value of $\frac{(a+x)(b+x)}{c+x}$.

Put $c+x=y$; then

$$\text{the expression} = \frac{(a-c+y)(b-c+y)}{y}$$

$$= \frac{(a-c)(b-c)}{y} + y + a - c + b - c$$

$$= \left(\frac{\sqrt{(a-c)(b-c)}}{\sqrt{y}} - \sqrt{y}\right)^2 + a - c + b - c + 2\sqrt{(a-c)(b-c)}.$$

Hence the expression is a minimum when the square term is zero; that is when $y=\sqrt{(a-c)(b-c)}$.

Thus the minimum value is

$$a-c+b-c+2\sqrt{(a-c)(b-c)};$$

and the corresponding value of x is $\sqrt{(a-c)(b-c)}-c$.

EXAMPLES. XIX. a.

1. Prove that $(ab+xy)(ax+by)>4abxy$.

2. Prove that $(b+c)(c+a)(a+b)>8abc$.

3. Shew that the sum of any real positive quantity and its reciprocal is never less than 2.

4. If $a^2+b^2=1$, and $x^2+y^2=1$, shew that $ax+by<1$.

5. If $a^2+b^2+c^2=1$, and $x^2+y^2+z^2=1$, shew that
$$ax+by+cz<1.$$

6. If $a>b$, shew that $a^ab^b>a^bb^a$, and $\log\frac{b}{a}<\log\frac{1+b}{1+a}$.

7. Shew that $(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)>9x^2y^2z^2$.

8. Find which is the greater $3ab^2$ or a^3+2b^3.

9. Prove that $a^3b+ab^3<a^4+b^4$.

10. Prove that $6abc<bc(b+c)+ca(c+a)+ab(a+b)$.

11. Shew that $b^2c^2+c^2a^2+a^2b^2>abc(a+b+c)$.

12. Which is the greater x^3 or x^2+x+2 for positive values of x?

13. Shew that $x^3+13a^2x > 5ax^2+9a^3$, if $x > a$.

14. Find the greatest value of x in order that $7x^2+11$ may be greater than x^3+17x.

15. Find the minimum value of $x^2-12x+40$, and the maximum value of $24x-8-9x^2$.

16. Shew that $(\lfloor n)^2 > n^n$, and $2.4.6...2n < (n+1)^n$.

17. Shew that $(x+y+z)^3 > 27xyz$.

18. Shew that $n^n > 1.3.5...(2n-1)$.

19. If n be a positive integer greater than 2, shew that

$$2^n > 1+n\sqrt{2^{n-1}}.$$

20. Shew that $(\lfloor n)^3 < n^n \left(\frac{n+1}{2}\right)^{2n}$.

21. Shew that

(1) $(x+y+z)^3 > 27(y+z-x)(z+x-y)(x+y-z)$.

(2) $xyz > (y+z-x)(z+x-y)(x+y-z)$.

22. Find the maximum value of $(7-x)^4(2+x)^5$ when x lies between 7 and -2.

23. Find the minimum value of $\frac{(5+x)(2+x)}{1+x}$.

*257. *To prove that if* a *and* b *are positive and unequal,* $\frac{a^m+b^m}{2} > \left(\frac{a+b}{2}\right)^m$, *except when* m *is a positive proper fraction.*

We have $a^m+b^m = \left(\frac{a+b}{2}+\frac{a-b}{2}\right)^m + \left(\frac{a+b}{2}-\frac{a-b}{2}\right)^m$; and since $\frac{a-b}{2}$ is less than $\frac{a+b}{2}$, we may expand each of these expressions in ascending powers of $\frac{a-b}{2}$. [Art. 184.]

$$\therefore \frac{a^m+b^m}{2} = \left(\frac{a+b}{2}\right)^m + \frac{m(m-1)}{1.2}\left(\frac{a+b}{2}\right)^{m-2}\left(\frac{a-b}{2}\right)^2$$

$$+\frac{m(m-1)(m-2)(m-3)}{1.2.3.4}\left(\frac{a+b}{2}\right)^{m-4}\left(\frac{a-b}{2}\right)^4+\ldots.$$

(1) If m is a positive integer, or any negative quantity, all the terms on the right are positive, and therefore

$$\frac{a^m + b^m}{2} > \left(\frac{a+b}{2}\right)^m.$$

(2) If m is positive and less than 1, all the terms on the right after the first are negative, and therefore

$$\frac{a^m + b^m}{2} < \left(\frac{a+b}{2}\right)^m.$$

(3) If $m > 1$ and positive, put $m = \frac{1}{n}$ where $n < 1$; then

$$\left(\frac{a^m + b^m}{2}\right)^{\frac{1}{m}} = \left(\frac{a^{\frac{1}{n}} + b^{\frac{1}{n}}}{2}\right)^n;$$

$$\therefore \left(\frac{a^m + b^m}{2}\right)^{\frac{1}{m}} > \frac{(a^{\frac{1}{n}})^n + (b^{\frac{1}{n}})^n}{2}, \text{ by (2)};$$

$$\therefore \left(\frac{a^m + b^m}{2}\right)^{\frac{1}{m}} > \frac{a+b}{2}.$$

$$\therefore \frac{a^m + b^m}{2} > \left(\frac{a+b}{2}\right)^m.$$

Hence the proposition is established. If $m = 0$, or 1, the inequality becomes an equality.

*258. *If there are* n *positive quantities* a, b, c,...k, *then*

$$\frac{a^m + b^m + c^m + \ldots + k^m}{n} > \left(\frac{a+b+c+\ldots+k}{n}\right)^m$$

unless m *is a positive proper fraction.*

Suppose m to have any value not lying between 0 and 1.

Consider the expression $a^m + b^m + c^m + \ldots + k^m$, and suppose that a and b are unequal; if we replace a and b by the two equal quantities $\frac{a+b}{2}$, $\frac{a+b}{2}$, the value of $a + b + c + \ldots + k$ remains unaltered, but the value of $a^m + b^m + c^m + \ldots + k^m$ is diminished, since

$$a^m + b^m > 2\left(\frac{a+b}{2}\right)^m.$$

Hence so long as any two of the quantities $a, b, c, \ldots k$ are unequal the expression $a^m + b^m + c^m + \ldots + k^m$ can be diminished without altering the value of $a + b + c + \ldots + k$; and therefore the value of $a^m + b^m + c^m + \ldots + k^m$ will be least when all the quantities $a, b, c, \ldots k$ are equal. In this case each of the quantities is equal to

$$\frac{a + b + c + \ldots + k}{n};$$

and the value of $a^m + b^m + c^m + \ldots + k^m$ then becomes

$$n\left(\frac{a + b + c + \ldots + k}{n}\right)^m.$$

Hence when $a, b, c, \ldots k$ are unequal,

$$\frac{a^m + b^m + c^m + \ldots + k^m}{n} > \left(\frac{a + b + c + \ldots + k}{n}\right)^m.$$

If m lies between 0 and 1 we may in a similar manner prove that the sign of inequality in the above result must be reversed.

The proposition may be stated verbally as follows:

The arithmetic mean of the m[th] *powers of* n *positive quantities is greater than the* m[th] *power of their arithmetic mean in all cases except when* m *lies between* 0 *and* 1.

*259. *If* a *and* b *are positive integers, and* a > b, *and if* x *be a positive quantity,*

$$\left(1 + \frac{x}{a}\right)^a > \left(1 + \frac{x}{b}\right)^b.$$

For

$$\left(1 + \frac{x}{a}\right)^a = 1 + x + \left(1 - \frac{1}{a}\right)\frac{x^2}{\underline{|2}} + \left(1 - \frac{1}{a}\right)\left(1 - \frac{2}{a}\right)\frac{x^3}{\underline{|3}} + \ldots (1),$$

the series consisting of $a + 1$ terms; and

$$\left(1 + \frac{x}{b}\right)^b = 1 + x + \left(1 - \frac{1}{b}\right)\frac{x^2}{\underline{|2}} + \left(1 - \frac{1}{b}\right)\left(1 - \frac{2}{b}\right)\frac{x^3}{\underline{|3}} + \ldots (2),$$

the series consisting of $b + 1$ terms.

After the second term, each term of (1) is greater than the corresponding term of (2); moreover the number of terms in (1) is greater than the number of terms in (2); hence the proposition is established.

*260. *To prove that* $\sqrt[x]{\dfrac{1+x}{1-x}} > \sqrt[y]{\dfrac{1+y}{1-y}}$,
if x *and* y *are proper fractions and positive, and* x > y.

For
$$\sqrt[x]{\frac{1+x}{1-x}} > \text{ or } < \sqrt[y]{\frac{1+y}{1-y}},$$
according as
$$\frac{1}{x}\log\frac{1+x}{1-x} > \text{ or } < \frac{1}{y}\log\frac{1+y}{1-y}.$$

But
$$\frac{1}{x}\log\frac{1+x}{1-x} = 2\left(1+\frac{x^2}{3}+\frac{x^4}{5}+\dots\right), \text{ [Art. 226]};$$
and
$$\frac{1}{y}\log\frac{1+y}{1-y} = 2\left(1+\frac{y^2}{3}+\frac{y^4}{5}+\dots\right).$$
$$\therefore \quad \frac{1}{x}\log\frac{1+x}{1-x} > \frac{1}{y}\log\frac{1+y}{1-y},$$
and thus the proposition is proved.

*261. *To prove that* $(1+x)^{1+x}(1-x)^{1-x} > 1$, *if* $x < 1$, *and to deduce that*
$$a^a b^b > \left(\frac{a+b}{2}\right)^{a+b}.$$

Denote $(1+x)^{1+x}(1-x)^{1-x}$ by P; then
$$\log P = (1+x)\log(1+x) + (1-x)\log(1-x)$$
$$= x\{\log(1+x) - \log(1-x)\} + \log(1+x) + \log(1-x)$$
$$= 2x\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\right) - 2\left(\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\dots\right)$$
$$= 2\left(\frac{x^2}{1\,.\,2}+\frac{x^4}{3\,.\,4}+\frac{x^6}{5\,.\,6}+\dots\right).$$

Hence $\log P$ is positive, and therefore $P > 1$;
that is,
$$(1+x)^{1+x}(1-x)^{1-x} > 1.$$

In this result put $x=\dfrac{z}{u}$, where $u>z$; then

$$\left(1+\frac{z}{u}\right)^{1+\frac{z}{u}}\left(1-\frac{z}{u}\right)^{1-\frac{z}{u}}>1;$$

$$\therefore \quad \left(\frac{u+z}{u}\right)^{u+z}\left(\frac{u-z}{u}\right)^{u-z}>1^u, \text{ or } 1;$$

$$\therefore \quad (u+z)^{u+z}(u-z)^{u-z}>u^{2u}.$$

Now put $u+z=a$, $u-z=b$, so that $u=\dfrac{a+b}{2}$;

$$\therefore \quad a^a b^b>\left(\frac{a+b}{2}\right)^{a+b}.$$

*EXAMPLES. XIX. b.

1. Shew that $27(a^4+b^4+c^4)>(a+b+c)^4$.

2. Shew that $n(n+1)^3<8(1^3+2^3+3^3+\ldots+n^3)$.

3. Shew that the sum of the m^{th} powers of the first n even numbers is greater than $n(n+1)^m$, if $m>1$.

4. If α and β are positive quantities, and $\alpha>\beta$, shew that

$$\left(1+\frac{1}{\alpha}\right)^{\alpha}>\left(1+\frac{1}{\beta}\right)^{\beta}.$$

Hence shew that if $n>1$ the value of $\left(1+\dfrac{1}{n}\right)^n$ lies between 2 and 2·718...

5. If a, b, c are in descending order of magnitude, shew that

$$\left(\frac{a+c}{a-c}\right)^a<\left(\frac{b+c}{b-c}\right)^b.$$

6. Shew that $\left(\dfrac{a+b+c+\ldots+k}{n}\right)^{a+b+c+\ldots+k}<a^a b^b c^c\ldots k^k$.

7. Prove that $\dfrac{1}{m}\log(1+a^m)<\dfrac{1}{n}\log(1+a^n)$, if $m>n$.

8. If n is a positive integer and $x<1$, shew that

$$\frac{1-x^{n+1}}{n+1}<\frac{1-x^n}{n}.$$

9. If a, b, c are in H. P. and $n > 1$, shew that $a^n + c^n > 2b^n$.

10. Find the maximum value of $x^3(4a-x)^5$ if x is positive and less than $4a$; and the maximum value of $x^{\frac{1}{2}}(1-x)^{\frac{1}{3}}$ when x is a proper fraction.

11. If x is positive, shew that $\log(1+x) < x$ and $> \dfrac{x}{1+x}$.

12. If $x+y+z=1$, shew that the least value of $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}$ is 9; and that $(1-x)(1-y)(1-z) > 8xyz$.

13. Shew that $(a+b+c+d)(a^3+b^3+c^3+d^3) > (a^2+b^2+c^2+d^2)^2$.

14. Shew that the expressions

$$a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)$$

and

$$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)$$

are both positive.

15. Shew that $(x^m+y^m)^n < (x^n+y^n)^m$, if $m > n$.

16. Shew that $a^b b^a < \left(\dfrac{a+b}{2}\right)^{a+b}$.

17. If a, b, c denote the sides of a triangle, shew that

(1) $a^2(p-q)(p-r)+b^2(q-r)(q-p)+c^2(r-p)(r-q)$

cannot be negative; p, q, r being any real quantities;

(2) $a^2yz+b^2zx+c^2xy$ cannot be positive, if $x+y+z=0$.

18. Shew that $\underline{|1}\ \underline{|3}\ \underline{|5}\ \dots\dots\dots\ \underline{|2n-1} > (\underline{|n})^n$.

19. If a, b, c, d, ... are p positive integers, whose sum is equal to n, shew that the least value of

$$\underline{|a}\,\underline{|b}\,\underline{|c}\,\underline{|d}\ \dots\dots\dots \text{ is } (\underline{|q})^{p-r}(\underline{|q+1})^r,$$

where q is the quotient and r the remainder when n is divided by p.

Questions on Quadratics, Dep. Exam Paper

1404, (a). If m & n are roots of $ax^2+bx+c=0$ prove $ax^2+bx+c = a(x-m)(x-n)$.

(b). State & prove converse.

(c). Construct equation whose roots are reciprocals of roots of $17x^2+53x-97=0$

(d). If x^2+6x-b, $x^2+12x+3b$ have a common linear factor, what values can b have and what is the factor in each case.

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(a). Form quadratic equation, sum roots $=A$. Sum of cubes $=B^3$.

(b). If p and q are roots of $C(x^2+m^2)+Cmx+Dm^2x^2=0$.

prove, $C(p^2+q^2)+Cpq+Dp^2q^2=0$.

1402. For what values of x is the expression $65x^2-241x+86$ equal to zero.

For what values is it positive?

" " " " negative?

1403. If $\left.\begin{array}{l} ax^3+2bx+c=0 \\ px^2+2qx+r=0 \end{array}\right\}$ have a common root.

what relations ~~does~~ connect a, b, c, p, q, r?

What relations when 2 roots are common.

if x has only one value.

1813. If the equations $ax + hy + gz = 0$
$hx + by + fz = 0$
$gx + fy + cz = 0$
are consistent, then
$ax^2 + 2hxy + by^2 + 2gzx + 2fyz + cz^2$
has 2 rational linear factors.

ANSWERS.

I. Pages 10—12.

1. (1) $54b : a$. (2) $9 : 7$. (3) $bx : ay$. **2.** 18. **3.** 385, 660.
4. 11. **5.** $5 : 13$. **6.** $5 : 6$ or $-3 : 5$.

10. $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$, or $\frac{x}{1} = \frac{y}{-1} = \frac{z}{0}$. **17.** $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

20. 3, 4, 1. **21.** -3, 4, 1. **22.** 7, 3, 2. **23.** 3, 4, 1.

25. $\pm a(b^2 - c^2)$, $\pm b(c^2 - a^2)$, $\pm c(a^2 - b^2)$.

26. $bc(b - c)$, $ca(c - a)$, $ab(a - b)$.

II. Pages 19, 20.

1. 45. **2.** (1) 12. (2) $300a^3b$. **3.** $\frac{x^3}{y(x^2 + y^2)}$.

13. 0, 5, $\frac{8}{7}$. **14.** 0, 3, 8. **15.** $\frac{a(b + c)}{cm - bm - 2an}$.

18. 8. **19.** 6, 9, 10, 15. **20.** 3 gallons from A; 8 gallons from B.
21. 45 gallons. **23.** $17 : 3$. **24.** $a = 4b$.
25. 64 per cent. copper and 36 per cent. zinc. 3 parts of brass are taken to 5 parts of bronze. **26.** 63 or 12 minutes.

III. Pages 26, 27.

1. $5\frac{1}{3}$. **2.** 9. **3.** $1\frac{1}{3}$. **4.** 2. **7.** 60.

9. $y = 2x - \frac{8}{x}$. **10.** $y = 5x + \frac{36}{x^2}$. **11.** 4.

12. $x = \frac{22}{15}z + \frac{2}{15z}$. **14.** 36. **15.** 1610 feet; 305·9 feet.

16. $22\frac{11}{24}$ cubic feet. **17.** $4 : 3$.
18. The regatta lasted 6 days; 4th, 5th, 6th days.
20. 16, 25 years; £200, £250. **21.** 1 day 18 hours 28 minutes.
22. The cost is least when the rate is 12 miles an hour; and then the cost per mile is £$\frac{3}{32}$, and for the journey is £9. 7s. 6d.

IV. a. Pages 31, 32.

1. $277\frac{1}{2}$. **2.** 153. **3.** 0. **4.** $\frac{n(10-n)}{3}$. **5.** 30.

6. -42. **7.** -185. **8.** $1325\sqrt{3}$. **9.** $75\sqrt{5}$.

10. $820a-1680b$. **11.** $n(n+1)a-n^2b$. **12.** $\frac{21}{2}(11a-9b)$.

13. $-\frac{1}{4}, -\frac{3}{4}, \ldots, -9\frac{1}{4}$. **14.** $1, -1\frac{1}{2}, \ldots, -39$. **15.** $-33x, -31x, \ldots, x$.

16. $x^2-x+1,\ x^2-2x+2, \ldots, x$. **17.** n^2. **18.** 3. **19.** 5.

20. 612. **21.** 4, 9, 14. **22.** 1, 4, 7. **23.** 495. **24.** 160.

25. $\frac{p(p+1)}{2a}+pb$. **26.** $n(n+1)a-\frac{n^2}{a}$.

IV. b. Pages 35, 36.

1. 10 or -8. **2.** 8 or -13. **3.** 2, 5, 8,...

4. First term 8, number of terms 59.

5. First term $7\frac{1}{2}$, number of terms 54.

6. Instalments £51, £53, £55,... **7.** 12. **8.** 25.

9. $\frac{n}{2(1-x)}(2+\overline{n-3}\,.\sqrt{x})$. **10.** n^2. **12.** $-(p+q)$.

13. 3, 5, 7, 9. [Assume for the numbers $a-3d$, $a-d$, $a+d$, $a+3d$.]

14. 2, 4, 6, 8. **15.** $p+q-m$. **16.** 12 or -17. **17.** $6r-1$.

20. $10p-8$. **21.** 8 terms. Series $1\frac{1}{2}, 3, 4\frac{1}{2}, \ldots\ldots$

22. 3, 5, 7; 4, 5, 6. **23.** $ry=(n+1-r)x$.

V. a. Pages 41, 42.

1. $\frac{2059}{1458}$. **2.** $\frac{1281}{512}$. **3.** $191\frac{1}{4}$. **4.** -682.

5. $\frac{1093}{45}$. **6.** $\frac{1}{4}(5^p-1)$. **7.** $\frac{9}{7}\left\{1-\left(\frac{4}{3}\right)^{2n}\right\}$. **8.** $364(\sqrt{3}+1)$.

9. $\frac{1}{2}(585\sqrt{2}-292)$. **10.** $-\frac{463}{192}$. **11.** $\frac{3}{2}, 1, \frac{2}{3}$.

12. $\frac{16}{3}, 8, \ldots, 27$. **13.** $-7, \frac{7}{2}, \ldots, \frac{7}{32}$. **14.** $\frac{64}{65}$.

15. $\frac{27}{58}$. **16.** $\cdot 999$. **17.** $\frac{1}{2}$. **18.** $\frac{3(3+\sqrt{3})}{2}$.

19. $7(7+\sqrt{42})$. **20.** 2. **21.** 16, 24, 36,... **22.** 2.

23. 2. **24.** 8, 12, 18. **25.** 2, 6, 18. **28.** $6, -3, 1\frac{1}{2}, \ldots\ldots$

V. b. Pages 45, 46.

1. $\dfrac{1-a^n}{(1-a)^2} - \dfrac{na^n}{1-a}$. **2.** $\dfrac{8}{3}$. **3.** $\dfrac{1-x}{(1-x)^2}$.

4. $4 - \dfrac{1}{2^{n-2}} - \dfrac{n}{2^{n-1}}$. **5.** 6. **6.** $\dfrac{1}{(1-x)^3}$.

9. $\dfrac{1}{(1-r)(1-br)}$. **10.** 40, 20, 10. **11.** $4, 1, \dfrac{1}{4}, \ldots$

12. $\dfrac{x(x^n-1)}{x-1} + \dfrac{n(n+1)a}{2}$. **13.** $\dfrac{x^2(x^{2n}-1)}{x^2-1} + \dfrac{xy(x^ny^n-1)}{xy-1}$.

14. $4p^2a + \dfrac{2}{9}\left(1 - \dfrac{1}{2^{2p}}\right)$. **15.** $1\frac{1}{5}$. **16.** $\dfrac{23}{48}$. **19.** $n \cdot 2^{n+2} - 2^{n+1} + 2$.

20. $\dfrac{(1+a)(a^nc^n-1)}{ac-1}$. **21.** $\dfrac{a}{r-1}\left\{\dfrac{r(r^{2n}-1)}{r^2-1} - n\right\}$.

VI. a. Pages 52, 53.

1. (1) 5. (2) $3\frac{1}{2}$. (3) $3\frac{23}{32}$. **2.** $6\frac{1}{5}, 7\frac{6}{7}$. **3.** $\dfrac{2}{5}, \dfrac{2}{7}, \dfrac{2}{9}, \dfrac{2}{11}$.

4. 6 and 24. **5.** 4 : 9. **10.** $n^2(n+1)$.

11. $\dfrac{1}{4}n(n+1)(n^2+n+3)$. **12.** $\dfrac{1}{6}n(n+1)(2n+7)$.

13. $\dfrac{1}{2}n(n+1)(n^2+3n+1)$. **14.** $\dfrac{1}{2}(3^{n+1}+1) - 2^{n+1}$.

15. $4^{n+1} - 4 - n(n+1)(n^2-n-1)$.

18. The n^{th} term $= b + c(2n-1)$, for all values of n greater than 1. The first term is $a+b+c$; the other terms form the A.P. $b+3c, b+5c, b+7c, \ldots$.

19. n^4. **22.** $\dfrac{n}{2}(2a+\overline{n-1}d)\left\{a^2+(n-1)ad+\dfrac{n(n-1)}{2}d^2\right\}$.

VI. b. Page 56.

1. 1240. **2.** 1140. **3.** 16646. **4.** 2470. **5.** 21321.
6. 52. **7.** 11879. **8.** 1840. **9.** 11940. **10.** 190.
11. 300. **12.** 18296. **14.** Triangular 364; Square 4900.
15. 120. **16.** $n-1$.

VII. a. Page 59.

1. 333244. **2.** 728626. **3.** 1740137. **4.** *e*7074. **5.** 112022.
6. 334345. **7.** 17832126. **8.** 1625. **9.** 2012. **10.** 342.
11. *ttt*90001. **12.** 231. **13.** 1456. **14.** 7071. **15.** *eee*.
16. (1) 121. (2) 122000.

VII. b. Pages 65, 66.

1. 20305. **2.** 4444. **3.** 11001110. **4.** 2000000. **5.** 7338.
6. 34402. **7.** 6587. **8.** 8978. **9.** 26011. **10.** 37214.

11. 30034342. **12.** $710te3$. **13.** 2714687. **14.** $\cdot\dot{2}04\dot{6}$. **15.** $15\cdot 1t6$.

16. $20\cdot 73$. **17.** $125\cdot 01\dot{2}\dot{5}$. **18.** $\frac{5}{8}$. **19.** $\frac{2}{3}$, $\frac{5}{8}$.

20. Nine. **21.** Four. **22.** Twelve. **23.** Eight. **24.** Eleven.

25. Twelve. **26.** Ten. **30.** $2^{11}+2^7+2^6$.

31. $3^9-3^8-3^7-3^6 \quad 3^5+3^3+3^2+1$.

VIII. a. PAGES 72, 73.

1. $\dfrac{2+\sqrt{2}+\sqrt{6}}{4}$. **2.** $\dfrac{3+\sqrt{6}+\sqrt{15}}{6}$.

3. $\dfrac{a\sqrt{b}+b\sqrt{a}-\sqrt{ab(a+b)}}{2ab}$. **4.** $\dfrac{a-1+\sqrt{a^2-1}+\sqrt{2a(a-1)}}{a-1}$.

5. $\dfrac{3\sqrt{30}+5\sqrt{15}-12-10\sqrt{2}}{7}$. **6.** $\dfrac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{2}$.

7. $3^{\frac{5}{3}}+3^{\frac{4}{3}}\cdot 2^{\frac{1}{2}}+3\cdot 2+3^{\frac{2}{3}}\cdot 2^{\frac{3}{2}}+3^{\frac{1}{3}}\cdot 2^2+2^{\frac{5}{2}}$.

8. $5^{\frac{5}{6}}-5^{\frac{4}{6}}\cdot 2^{\frac{1}{3}}+5^{\frac{3}{6}}\cdot 2^{\frac{2}{3}}-5^{\frac{2}{6}}\cdot 2+5^{\frac{1}{6}}\cdot 2^{\frac{4}{3}}-2^{\frac{5}{3}}$.

9. $a^{\frac{11}{6}}-a^{\frac{10}{6}}b^{\frac{1}{4}}+a^{\frac{9}{6}}b^{\frac{1}{2}}-\ldots+a^{\frac{1}{6}}b^{\frac{10}{4}}-b^{\frac{11}{4}}$. **10.** $3^{\frac{2}{3}}+3^{\frac{1}{3}}+1$.

11. $2^3-2^2\cdot 7^{\frac{1}{4}}+2\cdot 7^{\frac{1}{2}}-7^{\frac{3}{4}}$.

12. $5^{\frac{11}{3}}+5^{\frac{10}{3}}\cdot 3^{\frac{1}{4}}+5^3\cdot 3^{\frac{2}{4}}+\ldots+5^{\frac{1}{3}}\cdot 3^{\frac{10}{4}}+3^{\frac{11}{4}}$. **13.** $\dfrac{1-3^{\frac{2}{3}}+3^{\frac{1}{3}}}{2}$.

14. $17-3^{\frac{5}{3}}\cdot 2^{\frac{3}{2}}+3^{\frac{4}{3}}\cdot 2^2-3\cdot 2^{\frac{5}{2}}+3^{\frac{2}{3}}\cdot 2^3-3^{\frac{1}{3}}\cdot 2^{\frac{7}{2}}$.

15. $3^2\cdot 2^{\frac{1}{2}}-3^{\frac{5}{3}}\cdot 2+3^{\frac{4}{3}}\cdot 2^{\frac{3}{2}}-3\cdot 2^2+3^{\frac{2}{3}}\cdot 2^{\frac{5}{2}}-3^{\frac{1}{3}}\cdot 2^3$.

16. $\dfrac{1}{2}\left(3^{\frac{5}{6}}-3^{\frac{4}{6}}+3^{\frac{3}{6}}-3^{\frac{2}{6}}+3^{\frac{1}{6}}-1\right)$. **17.** $\dfrac{2^5+2^{\frac{31}{6}}+2^{\frac{26}{6}}+2^{\frac{21}{6}}+2^{\frac{16}{6}}+2^{\frac{11}{6}}+1}{31}$.

18. $\dfrac{3^{\frac{3}{2}}+3^{\frac{5}{6}}+3^{\frac{1}{6}}}{8}$. **19.** $\sqrt{5}+\sqrt{7}-2$.

20. $\sqrt{5}-\sqrt{7}+2\sqrt{3}$. **21.** $1+\sqrt{3}-\sqrt{2}$. **22.** $1+\sqrt{\frac{3}{2}}-\sqrt{\frac{5}{2}}$.

23. $2+\sqrt{a}-\sqrt{3b}$. **24.** $3-\sqrt{7}+\sqrt{2}-\sqrt{3}$. **25.** $1+\sqrt{3}$.

26. $2+\sqrt{5}$. **27.** $3-2\sqrt{2}$. **28.** $\sqrt{14}-2\sqrt{2}$.

29. $2\sqrt{3}+\sqrt{5}$. **30.** $3\sqrt{3}-\sqrt{6}$. **31.** $\sqrt{\dfrac{2a+x}{2}}+\sqrt{\dfrac{x}{2}}$.

32. $\sqrt{\dfrac{3a+b}{2}}-\sqrt{\dfrac{a-b}{2}}$. **33.** $\sqrt{\dfrac{1+a+a^2}{2}}+\sqrt{\dfrac{1-a+a^2}{2}}$.

34. $\dfrac{1}{\sqrt[4]{1-a^2}}\left(\sqrt{\dfrac{1+a}{2}}+\sqrt{\dfrac{1-a}{2}}\right)$.

35. $11+56\sqrt{3}$. **36.** 289. **37.** $\frac{1}{3}\sqrt{3}$.

38. $3\sqrt{3}+5$. 39. 3. 40. $8\sqrt{3}$.
41. $3+\sqrt{5}=5\cdot 23607$. 42. $x^2+1+\sqrt[3]{4}+x-x\sqrt[3]{2}+\sqrt[3]{2}$.
43. $3a+\sqrt{b^2-3a^2}$. 44. $\dfrac{a-1}{2}$.

VIII. b. Pages 81, 82.

1. $6-2\sqrt{6}$. 2. -13. 3. $e^{2\sqrt{-1}}-e^{-2\sqrt{-1}}$.
4. x^2-x+1. 5. $\dfrac{3+\sqrt{-2}}{11}$. 6. $-19-6\sqrt{10}$.
7. $-\dfrac{8}{29}$. 8. $\dfrac{4ax\sqrt{-1}}{a^2+x^2}$. 9. $\dfrac{2(3x^2-1)\sqrt{-1}}{x^2+1}$.
10. $\dfrac{3a^2-1}{2a}$. 11. $\sqrt{-1}$. 12. 100.
13. $\pm(2+3\sqrt{-1})$. 14. $\pm(5-6\sqrt{-1})$. 15. $\pm(1+4\sqrt{-3})$.
16. $\pm 2(1-\sqrt{-1})$. 17. $\pm(a+\sqrt{-1})$. 18. $\pm\{(a+b)-(a-b)\sqrt{-1}\}$.
19. $-\dfrac{9}{13}+\dfrac{19}{13}i$. 20. $\dfrac{4}{7}-\dfrac{\sqrt{6}}{14}i$. 21. i.
22. $-\dfrac{1}{5}+\dfrac{3}{5}i$. 23. $\dfrac{2b(3a^2-b^2)}{a^2+b^2}i$.

IX. a. Pages 88—90.

1. $35x^2+13x-12=0$. 2. $mnx^2+(n^2-m^2)x-mn=0$.
3. $(p^2-q^2)x^2+4pqx-p^2+q^2=0$. 4. $x^2-14x+29=0$.
5. $x^2+10x+13=0$. 6. $x^2+2px+p^2-8q=0$.
7. $x^2+6x+34=0$. 8. $x^2+2ax+a^2+b^2=0$.
9. $x^2+a^2-2ab+b^2=0$. 10. $6x^3+11x^2-19x+6=0$.
11. $2ax^3+(4-a^2)x^2-2ax=0$. 12. $x^3-8x^2+17x-4=0$.
14. 3, 5. 15. $2,\ -\dfrac{10}{9}$. 16. $\dfrac{a-b}{a+b}$.
18. $\dfrac{b^2-2ac}{c^2}$. 19. $\dfrac{bc^4(3ac-b^2)}{a^7}$. 20. $\dfrac{b^2(b^2-4ac)}{a^2c^4}$.
21. 7. 22. -15. 23. 0. 24. $x^2-2(p^2-2q)x+p^2(p^2-4q)=0$.
26. (1) $\dfrac{b^2-2ac}{a^2c^2}$. (2) $\dfrac{b(b^2-3ac)}{a^3c^3}$. 27. $nb^2=(1+n)^2ac$.
28. $a^2c^2x^2-(b^2-2ac)(a^2+c^2)x+(b^2-2ac)^2=0$.
29. $x^2-4mnx-(m^2-n^2)^2=0$.

IX. b. Pages 92, 93.

1. 2 and -2. 5. $bx^2-2ax+a=0$.
6. (1) $\dfrac{p(p^2-4q)(p^2-q)}{q}$. (2) $\dfrac{p^4-4p^2q+2q^2}{q^4}$. 11. $\dfrac{1}{3}$.

IX. c. Page 96.

1. -2. **2.** ± 7. **5.** $(ln'-l'n)^2=(lm'-l'm)(mn'-m'n)$.
7. $(aa'-bb')^2+4(ha'+h'b)(hb'+h'a)=0$.
10. $(bb'-2ac'-2a'c)^2=(b^2-4ac)(b'^2-4a'c')$; which reduces to $(ac'-a'c)^2=(ab'-a'b)(bc'-b'c)$.

X. a. Pages 101, 102.

1. $\frac{1}{4}, -\frac{1}{2}$. **2.** $+\frac{1}{3}, +1$. **3.** $4, \frac{1}{4}$. **4.** $\frac{4}{9}, \frac{1}{4}$.

5. $3^n, 2^n$. **6.** $1, 2^{2n}$. **7.** $27, \frac{25}{147}$. **8.** $\frac{9}{13}, \frac{4}{13}$.

9. $\frac{1}{9}, \frac{25}{4}$. **10.** $-1, -\frac{1}{32}$. **11.** $2, 0$. **12.** ± 1.

13. -4. **14.** $+3$. **15.** 0. **16.** $\frac{1}{8}, 450$.

17. $9, -7, 1\pm\sqrt{-24}$. **18.** $2, -4, -1\pm\sqrt{71}$. **19.** $3, -\frac{3}{2}, \frac{3\pm\sqrt{-47}}{4}$.

20. $4, -\frac{7}{2}, \frac{1\pm\sqrt{65}}{4}$. **21.** $2, -8, -3\pm 3\sqrt{5}$. **22.** $3, -\frac{5}{3}, \frac{2\pm\sqrt{70}}{3}$.

23. $5, \frac{1}{3}, \frac{8\pm\sqrt{148}}{3}$. **24.** $7, -\frac{14}{3}, \frac{7\pm\sqrt{37}}{6}$. **25.** $2, \frac{1}{2}, \frac{5\pm\sqrt{201}}{4}$.

26. $5, -\frac{7}{3}, \frac{8\pm\sqrt{415}}{6}$. **27.** $1, 3$. **28.** $5, \frac{1}{2}$.

29. $1, 9, -\frac{18}{5}$. **30.** $a, \frac{a}{2}, -\frac{a}{3}$. **31.** $2, -\frac{9}{2}$.

32. $4, -\frac{10}{3}$. **33.** $0, 5$. **34.** $6, -\frac{5}{2}$.

35. $1, \frac{-3\pm\sqrt{5}}{2}$. **36.** $3, \frac{1}{3}, \frac{-1\pm\sqrt{-35}}{6}$. **37.** $2\pm\sqrt{3}, \frac{-1\pm\sqrt{-3}}{2}$.

38. $2, -\frac{1}{2}, 5, -\frac{1}{5}$. **39.** $3a, -4a$. **40.** $\pm\frac{2a}{5}$.

41. $0, 1, 3$. **42.** $\frac{-1\pm\sqrt{17}}{2}, \frac{-1\pm\sqrt{2}}{2}$.

43. $\frac{3}{2}, \frac{2}{3}$. **44.** $3, -1$. **45.** ± 1.

46. 13. **47.** 4. **48.** $0, \frac{63a}{65}$.

49. $1, \frac{(\sqrt{a}-\sqrt{b})^2+4}{(\sqrt{a}+\sqrt{b})^2-4}$. **50.** ± 5. **51.** $5, -4, \frac{1\pm\sqrt{-75}}{2}$.

52. $-\frac{1}{3}, \frac{1\pm\sqrt{-31}}{6}$.

X. b. Pages 106, 107.

1. $x=5,\ -\frac{8}{3};\ y=4,\ -\frac{15}{2}.$

2. $x=2,\ -\frac{8}{19};\ y=7,\ -\frac{97}{19}.$

3. $x=1,\ -\frac{53}{88};\ y=1,\ -\frac{25}{22}.$

4. $x=\pm 5,\ \pm 3;\ y=\pm 3,\ \pm 5.$

5. $x=8,\ 2;\ y=2,\ 8.$

6. $x=45,\ 5;\ y=5,\ 45.$

7. $x=9,\ 4;\ y=4,\ 9.$

8. $x=\pm 2,\ \pm 3;\ y=\pm 1,\ \pm 2.$

9. $x=\pm 2,\ \pm 3;\ y=\pm 3,\ \pm 4.$

10. $x=\pm 5,\ \pm 3;\ y=\pm 3,\ \pm 4.$

11. $x=\pm 2,\ \pm 1;\ y=\pm 1,\ \pm 3.$

12. $x=\pm\sqrt{3},\ \pm\sqrt{\frac{3}{19}};\ y=0,\ \pm 6\sqrt{\frac{3}{19}}.$

13. $x=5,\ 3,\ 4\pm\sqrt{-97};\ y=3,\ 5,\ 4\mp\sqrt{-97}.$

14. $x=4,\ -2,\ \pm\sqrt{-15}+1;\ y=2,\ -4,\ \pm\sqrt{-15}-1.$

15. $x=4,\ -2,\ \pm\sqrt{-11}+1;\ y=2,\ -4,\ \pm\sqrt{-11}-1.$

16. $x=\frac{4}{5},\ \frac{1}{5};\ y=20,\ 5.$

17. $x=2,\ 1;\ y=1,\ 2.$

18. $x=6,\ 4;\ y=10,\ 15.$

19. $x=729,\ 343;\ y=343,\ 729.$

20. $x=16,\ 1;\ y=1,\ 16.$

21. $x=9,\ 4;\ y=4,\ 9.$

22. $x=5;\ y=\pm 4.$

23. $x=1,\ \frac{5}{3};\ y=2,\ \frac{2}{3}.$

24. $x=9,\ 1;\ y=1,\ 9.$

25. $x=\pm 25;\ y=\pm 9.$

26. $x=6,\ 2,\ 4,\ 3;\ y=1,\ 3,\ \frac{3}{2},\ 2.$

27. $x=\pm 5,\ \pm 4,\ \pm\frac{5}{2},\ \pm 2;\ y=\pm 5,\ \pm 4,\ \pm 10,\ \pm 8.$

28. $x=4,\ \frac{107}{13};\ y=1,\ \frac{48}{13}.$

29. $x=-6,\ \frac{1\pm\sqrt{-143}}{2};\ y=-3,\ \frac{1\pm 3\sqrt{-143}}{4}.$

30. $x=0,\ 9,\ 3;\ y=0,\ 3,\ 9.$

31. $x=0,\ 1,\ \frac{15}{22};\ y=0,\ 2,\ \frac{9}{22}.$

32. $x=5,\ \frac{10}{23},\ 0;\ y=3,\ -\frac{6}{23},\ -\frac{4}{7}.$

33. $x=2,\ \sqrt[3]{4},\ 2;\ y=2,\ 2\sqrt[3]{4},\ 6.$

34. $x=1,\ \sqrt[3]{\frac{1}{2}};\ y=2,\ 3\sqrt[3]{\frac{1}{2}}.$

35. $x=\pm 3,\ \pm\sqrt{-18};\ y=\pm 3,\ \mp\sqrt{-18}.$

36. $x=y=\pm 2.$

37. $x=0,\ \frac{b\sqrt{a}}{\sqrt{a}+\sqrt{b}},\ \frac{b\sqrt{a}}{\sqrt{a}-\sqrt{b}};\ y=0,\ \frac{a\sqrt{b}}{\sqrt{a}+\sqrt{b}},\ -\frac{a\sqrt{b}}{\sqrt{a}-\sqrt{b}}.$

38. $x=b,\ \frac{b(-1\pm\sqrt{3})}{2};\ y=a,\ a(1\mp\sqrt{3}).$

39. $x=\frac{a^2}{b},\ \frac{a(2b-a)}{b};\ y=\frac{b^2}{a},\ \frac{b(2a-b)}{a}.$

40. $x=0,\ \pm a\sqrt{7},\ \pm a\sqrt{13},\ \pm 3a,\ \pm a;\ y=0,\ \mp b\sqrt{7},\ \pm b\sqrt{13},\ \mp b,\ \mp 3b.$

41. $x=+1,\ \pm\dfrac{2a^2}{\sqrt{16a^4-a^2-1}};\ y=\pm 2a,\ \mp\dfrac{a}{\sqrt{16a^4-a^2-1}}.$

X. c. Pages 109, 110.

1. $x=\pm 3;\ y=\pm 5;\ z=\pm 4.$ **2.** $x=5;\ y=-1;\ z=7.$

3. $x=5,\ -1;\ y=1,\ -5;\ z=2.$ **4.** $x=8,\ -3;\ y=3;\ z=3,\ -8.$

5. $x=4,\ 3,\ \dfrac{2\pm\sqrt{151}}{3};\ y=3,\ 4,\ \dfrac{2\mp\sqrt{151}}{3};\ z=2,\ -\dfrac{11}{3}.$

6. $x=\pm 3;\ y=\mp 2;\ z=\pm 5.$ **7.** $x=\pm 5;\ y=\pm 1;\ z=\pm 1.$

8. $x=8,\ -8;\ y=5,\ -5;\ z=3,\ -3.$ **9.** $x=3;\ y=4;\ z=\dfrac{1}{2};\ u=\dfrac{1}{3}.$

10. $x=1;\ y=2;\ z=3.$ **11.** $x=5,\ -7;\ y=3,\ -5;\ z=6,\ -8.$

12. $x=1,\ -2;\ y=7,\ -3;\ z=3,\ -\dfrac{11}{3}.$

13. $x=4,\ \dfrac{60}{7};\ y=6,\ \dfrac{66}{7};\ z=2,\ -6.$ **14.** $x=a,\ 0,\ 0;\ y=0,\ a,\ 0;\ z=0,\ 0,\ a.$

15. $x=\dfrac{a}{\sqrt{3}},\ \dfrac{\sqrt{3}\pm\sqrt{-9}}{6}a;\ y=\dfrac{a}{\sqrt{3}},\ \dfrac{-5\sqrt{3}\pm\sqrt{-9}}{6}a;$

$z=\dfrac{a}{\sqrt{3}},\ -\dfrac{\sqrt{3}\pm\sqrt{-9}}{3}a.$

16. $x=a,\ -2a,\ \dfrac{7\pm\sqrt{-15}}{2}a;\ y=4a,\ a,\ \dfrac{-11\pm\sqrt{-15}}{2}a;$

$z=2a,\ -4a,\ (1\pm\sqrt{-15})\,a.$

X. d. Page 113.

1. $x=29,\ 21,\ 13,\ 5;\ y=2,\ 5,\ 8,\ 11.$

2. $x=1,\ 3,\ 5,\ 7,\ 9;\ y=24,\ 19,\ 14,\ 9,\ 4.$

3. $x=20,\ 8;\ y=1,\ 8.$ **4.** $x=9,\ 20,\ 31;\ y=27,\ 14,\ 1.$

5. $x=30,\ 5;\ y=9,\ 32.$ **6.** $x=50,\ 3;\ y=3,\ 44.$

7. $x=7p-5,\ 2;\ y=5p-4,\ 1.$ **8.** $x=13p-2,\ 11;\ y=6p-1,\ 5.$

9. $x=21p-9,\ 12;\ y=8p-5,\ 3.$ **10.** $x=17p,\ 17;\ y=13p,\ 13.$

11. $x=19p-16,\ 3;\ y=23p-19,\ 4.$ **12.** $x=77p-74,\ 3;\ y=30p-25,\ 5.$

13. 11 horses, 15 cows. **14.** 101. **15.** 56, 25 or 16, 65.

16. To pay 3 guineas and receive 21 half-crowns.

17. 1147; an infinite number of the form $1147+39\times 56p$.

18. To pay 17 florins and receive 3 half-crowns.

19. 37, 99; 77, 59; 117, 19.

20. 28 rams, 1 pig, 11 oxen; or 13 rams, 14 pigs, 13 oxen.

21. 3 sovereigns, 11 half-crowns, 13 shillings.

XI. a. Pages 122—124.

1. 12. **2.** 224. **3.** 40320, 6375600, 10626, 11628.
4. 6720. **5.** 15. **6.** 40320; 720. **7.** 15, 360.
8. 6. **9.** 120. **10.** 720. **11.** 10626, 1771.
12. 1440. **13.** 6375600. **14.** 360, 144. **15.** 230300.
16. 1140, 231. **17.** 144. **18.** 224, 896. **19.** 848.
20. 56. **21.** 360000. **22.** 2052000. **23.** 369600.
24. 21600. **25.** $\dfrac{\lfloor 45}{\lfloor 10 \, \lfloor 15 \, \lfloor 20}$. **26.** 2520. **27.** 5760.
28. 3456. **29.** 2903040. **30.** 25920. **32.** 41.
33. 1956. **34.** 7.

XI. b. Pages 131, 132.

1. (1) 1663200. (2) 129729600. (3) 3326400. **2.** 4084080.
3. 151351200. **4.** 360. **5.** 72. **6.** 125.
7. n^r. **8.** 531441. **9.** p^n. **10.** 30.
11. 1260. **12.** 3374. **13.** 455. **14.** $\dfrac{\lfloor a+2b+3c+d}{\lfloor a \, (\lfloor b)^2 \, (\lfloor c)^3}$.
15. 4095. **16.** 57760000. **17.** 1023. **18.** 720; 3628800.
19. 127. **20.** 315. **21.** $\dfrac{\lfloor mn}{(\lfloor m)^n \, \lfloor n}$. **22.** 64; 325. **23.** 42.
24. (1) $\dfrac{p(p-1)}{2} - \dfrac{q(q-1)}{2} + 1$; (2) $\dfrac{p(p-1)(p-2)}{6} - \dfrac{q(q-1)(q-2)}{6}$.
25. $\dfrac{p(p-1)(p-2)}{6} - \dfrac{q(q-1)(q-2)}{6} + 1$. **26.** $(p+1)^n - 1$.
27. 113; 2190. **28.** 2454. **29.** 6666600. **30.** 5199960.

XIII. a. Pages 142, 143.

1. $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$.
2. $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$.
3. $32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$.
4. $1 - 18a^2 + 135a^4 - 540a^6 + 1215a^8 - 1458a^{10} + 729a^{12}$.
5. $x^{10} + 5x^9 + 10x^8 + 10x^7 + 5x^6 + x^5$.
6. $1 - 7xy + 21x^2y^2 - 35x^3y^3 + 35x^4y^4 - 21x^5y^5 + 7x^6y^6 - x^7y^7$.
7. $16 - 48x^2 + 54x^4 - 27x^6 + \dfrac{81x^8}{16}$.
8. $729a^6 - 972a^5 + 540a^4 - 160a^3 + \dfrac{80a^2}{3} - \dfrac{64a}{27} + \dfrac{64}{729}$.
9. $1 + \dfrac{7x}{2} + \dfrac{21x^2}{4} + \dfrac{35x^3}{8} + \dfrac{35x^4}{16} + \dfrac{21x^5}{32} + \dfrac{7x^6}{64} + \dfrac{x^7}{128}$.

10. $\frac{64x^6}{729} - \frac{32x^4}{27} + \frac{20x^2}{3} - 20 + \frac{135}{4x^2} - \frac{243}{8x^4} + \frac{729}{64x^6}$.

11. $\frac{1}{256} + \frac{a}{16} + \frac{7a^2}{16} + \frac{7a^3}{4} + \frac{35a^4}{8} + 7a^5 + 7a^6 + 4a^7 + a^8$.

12. $1 - \frac{10}{x} + \frac{45}{x^2} - \frac{120}{x^3} + \frac{210}{x^4} - \frac{252}{x^5} + \frac{210}{x^6} - \frac{120}{x^7} + \frac{45}{x^8} - \frac{10}{x^9} + \frac{1}{x^{10}}$.

13. $-35750x^{10}$. **14.** $-112640x^9$. **15.** $-312x^2$.

16. $\frac{|\underline{30}}{|\underline{27}\,|\underline{3}}(5x)^3(8y)^{27}$. **17.** $40a^7b^3$. **18.** $\frac{1120}{81}a^4b^4$.

19. $\frac{10500}{x^3}$. **20.** $\frac{70x^6y^{10}}{a^2b^6}$. **21.** $2x^4 + 24x^2 + 8$.

22. $2x(16x^4 - 20x^2a^2 + 5a^4)$. **23.** $140\sqrt{2}$.

24. $2(365 - 363x + 63x^2 - x^3)$. **25.** 252. **26.** $-\frac{429}{16}x^{14}$.

27. $110565a^4$. **28.** $84a^3b^6$. **29.** $1365, -1365$.

30. $\frac{189a^{17}}{8}, -\frac{21}{16}a^{19}$. **31.** $\frac{7}{18}$. **32.** 18564.

33. $\frac{|\underline{n}}{|\underline{\tfrac{1}{2}(n-r)}\;|\underline{\tfrac{1}{2}(n+r)}}$. **34.** $(-1)^n\frac{|\underline{3n}}{|\underline{n}\,|\underline{2n}}$.

XIII. b. PAGES 147, 148.

1. The 9^{th}. **2.** The 12^{th}. **3.** The 6^{th}. **4.** The 10^{th} and 11^{th}.

5. The $3^{rd} = 6\frac{2}{3}$. **6.** The 4^{th} and $5^{th} = \frac{7}{144}$. **9.** $x = 2, y = 3, n = 5$.

10. $1 + 8x + 20x^2 + 8x^3 - 26x^4 - 8x^5 + 20x^6 - 8x^7 + x^8$.

11. $27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$.

12. $\frac{|\underline{n}}{|\underline{r-1}\;|\underline{n-r+1}}x^{r-1}a^{n-r+1}$. **13.** $(-1)^p\frac{|\underline{2n+1}}{|\underline{p+1}\;|\underline{2n-p}}x^{2p-2n+1}$.

14. 14. **15.** $2r = n$.

XIV. a. PAGE 155.

1. $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$. **2.** $1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3$.

3. $1 - \frac{2}{5}x - \frac{3}{25}x^2 - \frac{8}{125}x^3$. **4.** $1 - 2x^2 + 3x^4 - 4x^6$.

5. $1 - x - x^2 - \frac{5}{3}x^3$. **6.** $1 + x + 2x^2 + \frac{14}{3}x^3$.

7. $1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3$. **8.** $1 - x + \frac{2}{3}x^2 - \frac{10}{27}x^3$.

9. $1 + x + \frac{x^2}{6} - \frac{x^3}{54}$. **10.** $1 - 2a + \frac{5}{2}a^2 - \frac{5}{2}a^3$.

11. $\frac{1}{8}\left(1-\frac{3}{2}x+\frac{3}{2}x^2-\frac{5}{4}x^3\right)$.
12. $3\left(1+\frac{x}{9}-\frac{1}{162}x^2+\frac{1}{1458}x^3\right)$.

13. $4\left(1+a-\frac{1}{4}a^2+\frac{1}{6}a^3\right)$.
14. $\frac{1}{27}\left(1+x+\frac{5}{6}x^2+\frac{35}{54}x^3\right)$.

15. $\frac{1}{2a^{\frac{1}{2}}}\left(1+\frac{x}{a}+\frac{3}{2}\cdot\frac{x^2}{a^2}+\frac{5}{2}\cdot\frac{x^3}{a^3}\right)$.
16. $-\frac{429}{16}x^7$.
17. $\frac{77}{256}x^{30}$.

18. $-\frac{1040}{81}a^{18}$.
19. $\frac{16b^4}{243a^5}$.
20. $(r+1)x^r$.

21. $\frac{(r+1)(r+2)(r+3)}{1.2.3}x^r$.
22. $(-1)^{r-1}\frac{1.3.5\ldots(2r-3)}{2^r\,|\underline{r}}x^r$.

23. $(-1)^{r-4}\frac{11.8.5.2.1.4\ldots(3r-14)}{3^r\,|\underline{r}}x^r$.

24. $-1848x^{13}$.
25. $-\frac{19712}{3}x^6$.

XIV. b. Pages 161, 162.

1. $(-1)^r\frac{1.3.5.7\ldots(2r-1)}{2^r\,|\underline{r}}x^r$.
2. $\frac{(r+1)(r+2)(r+3)(r+4)}{|\underline{4}}x^r$.

3. $(-1)^{r-1}\frac{1.2.5\ldots(3r-4)}{|\underline{r}}x^r$.
4. $(-1)^r\frac{2.5.8\ldots(3r-1)}{3^r\,|\underline{r}}x^r$.

5. $(-1)^r\frac{(r+1)(r+2)}{|\underline{2}}x^{2r}$.
6. $\frac{3.5.7\ldots(2r+1)}{|\underline{r}}x^r$.

7. $(-1)^r\frac{b^r}{a^{r+1}}\cdot x^r$.
8. $\frac{r+1}{2^{r+2}}x^r$.

9. $-\frac{2.1.4\ldots(3r-5)}{3^r\,|\underline{r}}\cdot\frac{x^{3r}}{a^{3r-2}}$.
10. $(-1)^r\frac{1.3.5\ldots(2r-1)}{|\underline{r}}x^r$.

11. $\frac{2.5.8\ldots(3r-1)}{|\underline{r}}x^r$.
12. $\frac{(n+1)(2n+1)\ldots(\overline{r-1}.n+1)}{|\underline{r}}\cdot\frac{x^r}{a^{nr+1}}$.

13. The 3rd. 14. The 5th. 15. The 13th. 16. The 7th.
17. The 4th and 5th. 18. The 3rd. 19. 9·89949.
20. 9·99333. 21. 10·00999. 22. 6·99927. 23. ·19842.
24. 1·00133. 25. ·00795. 26. 5·00096. 27. $1-\frac{23x}{6}$.

28. $\frac{2}{3}\left(1+\frac{x}{24}\right)$.
29. $1-\frac{5x}{8}$.
30. $\frac{1}{4}-\frac{5}{6}x$.
31. $1-\frac{343}{120}x$.

32. $\frac{1}{3}-\frac{71}{360}x$.
35. $1-4x+13x^2$.
36. $2+\frac{29}{4}x+\frac{297}{32}x^2$.

XIV. c. Pages 167—169.

1. -197.
2. 142.
3. $(-1)^{n-1}$.

4. $(-1)^n(n^2+2n+2)$.
6. $\sqrt{8}=\left(1-\frac{1}{2}\right)^{-\frac{3}{2}}$.

7. $\left(1-\frac{2}{3}\right)^{-n}=2^n\left(1-\frac{1}{3}\right)^{-n}$. **12.** $\frac{\lfloor 2n}{\lfloor n \lfloor n}$.

14. Deduced from $(1-x^3)-(1-x)^3=3x-3x^2$. **16.** (1) 45. (2) 6561.

18. (1) Equate coefficients of x^r in $(1+x)^n(1+x)^{-1}=(1+x)^{n-1}$.

(2) Equate absolute terms in $(1+x)^n\left(1+\frac{1}{x}\right)^{-2}=x^2(1+x)^{n-2}$.

20. Series on the left $+(-1)^n q_n{}^2=$ coefficient of x^{2n} in $(1-x^2)^{-\frac{1}{2}}$.

21. $2^{2n-1}-\frac{1}{2}\cdot\frac{\lfloor 2n}{\lfloor n \lfloor n}$.

[Use $(c_0+c_1+c_2+\dots c_n)^2-2(c_0c_1+c_1c_2+\dots)=c_0{}^2+c_1{}^2+c_2{}^2+\dots c_n{}^2$].

XV. Pages 173, 174.

1. -12600. **2.** -168. **3.** 3360. **4.** $-1260a^2b^3c^4$.

5. -9. **6.** 8085. **7.** 30. **8.** 1905.

9. -10. **10.** $-\frac{3}{2}$. **11.** -1. **12.** $-\frac{4}{81}$.

13. $\frac{59}{16}$. **14.** -1. **15.** $\frac{211}{3}$. **16.** $1-\frac{1}{2}x-\frac{7}{8}x^2$.

17. $1-2x^2+4x^3+5x^4-20x^5$. **18.** $16\left(1-\frac{3}{2}x^3+3x^4+\frac{9}{32}x^6-\frac{9}{8}x^7+\frac{9}{8}x^8\right)$.

XVI. a. Pages 178, 179.

1. 8, 6. **2.** 2, -1. **3.** $-\frac{16}{3}$, $-\frac{1}{2}$. **4.** -4, $-\frac{3}{2}$.

5. $\frac{4}{3}$, $-\frac{4}{5}$. **6.** $\frac{2}{5}$, $-\frac{1}{2}$, $-\frac{5}{2}$. **7.** $\frac{7}{3}$, -3, $-\frac{4}{3}$, $\frac{2}{3}$.

8. $6\log a+9\log b$. **9.** $\frac{2}{3}\log a+\frac{3}{2}\log b$.

10. $-\frac{4}{9}\log a+\frac{1}{3}\log b$. **11.** $-\frac{2}{3}\log a-\frac{1}{2}\log b$.

12. $-\frac{7}{12}\log a-\log b$. **13.** $\frac{1}{2}\log a$. **14.** $-5\log c$. **16.** $\log 3$.

18. $\frac{\log c}{\log a-\log b}$. **19.** $\frac{5\log c}{2\log a+3\log b}$.

20. $\frac{\log a+\log b}{2\log c-\log a+\log b}$. **21.** $x=\frac{4\log m}{\log a}$, $y=-\frac{\log m}{\log b}$.

22. $\log x=\frac{1}{5}(a+3b)$, $\log y=\frac{1}{5}(a-2b)$. **24.** $\frac{\log(a-b)}{\log(a+b)}$.

XVI. b. Pages 185, 186.

1. 4, 1, 2, $\bar{2}$, $\bar{1}$, $\bar{1}$, $\bar{1}$.

2. ·8821259, 2·8821259, 3·8821259, 5·8821259, $\bar{6}$·8821259.

3. 5, 2, 4, 1.

4. Second decimal place; units' place; fifth decimal place.

5. 1·8061800. **6.** 1·9242793. **7.** $\bar{1}$·1072100. **8.** $\bar{2}$·0969100.

9. 1·1583626. **10.** ·6690067. **11.** ·3597271. **12.** ·0563520.

13. $\bar{1}$·5052973. **14.** ·44092388. **15.** 1·948445. **16.** 191563·1.

17. 1·1998692. **18.** 1·0039238. **19.** 9·076226. **20.** 178·141516.

21. 9. **23.** 301. **24.** 3·46. **25.** 4·29. **26.** 1·206. **27.** 14·206.

28. 4·562. **29.** $x=\dfrac{\log 3}{\log 3-\log 2}$; $y=\dfrac{\log 2}{\log 3-\log 2}$.

30. $x=\dfrac{3\log 3-2\log 2}{4(\log 3-\log 2)}$; $y=\dfrac{\log 3}{4(\log 3-\log 2)}$. **31.** 1·64601.

32. $\dfrac{\log 2}{2\log 7}=\cdot 1781$; $\dfrac{2\log 7}{\log 2}=5\cdot 614$.

XVII. Pages 195—197.

1. $\log_e 2$. **2.** $\log_e 3-\log_e 2$. **6.** ·0020000006666670.

9. $e^{x^2}-e^{y^2}$. **10.** ·8450980; 1·0413927; 1·1139434. In Art. 225 put $n=50$ in (2); $n=10$ in (1); and $n=1000$ in (1) respectively.

12. $(-1)^{r-1}\cdot\dfrac{2^r+1}{r}x^r$. **13.** $\dfrac{(-1)^{r-1}3^r+2^r}{r}x^r$.

14. $2\left\{1+\dfrac{(2x)^2}{\underline{|2}}+\dfrac{(2x)^4}{\underline{|4}}+\ldots+\dfrac{(2x)^{2r}}{\underline{|2r}}+\ldots\right\}$.

15. $1-\dfrac{x^2}{\underline{|2}}+\dfrac{x^4}{\underline{|4}}-\dfrac{x^6}{\underline{|6}}+\ldots+(-1)^r\dfrac{x^{2r}}{\underline{|2r}}+\ldots$ **18.** $\dfrac{x}{1-x}+\log_e(1-x)$.

24. ·69314718; 1·09861229; 1·60943792; $a=-\log_e\left(1-\dfrac{1}{10}\right)=\cdot 105360516$;

$$b=-\log_e\left(1-\frac{4}{100}\right)=\cdot 040821995;\quad c=\log_e\left(1+\frac{1}{80}\right)=\cdot 012422520.$$

XVIII. a. Page 202.

1. £1146. 14*s*. 10*d*. **2.** £720. **3.** 14·2 years.

4. £6768. 7*s*. $10\frac{1}{2}d$. **5.** 9·6 years. **8.** £496. 19*s*. $4\frac{3}{4}d$.

9. A little less than 7 years. **10.** £119. 18*s*. $5\frac{3}{4}d$.

XVIII. b. Page 207.

1. 6 per cent. **2.** £3137. 2*s*. $2\frac{2}{3}d$. **3.** £110.

4. 3 per cent. **5.** $28\frac{4}{7}$ years. **6.** £1275. **7.** £926. 2*s*.

8. £6755. 13*s*. **9.** £183. 18*s*. **10.** $3\frac{1}{3}$ per cent. **11.** £616. 9*s*. $1\frac{1}{2}d$.

13. £1308. 12*s*. $4\frac{1}{2}d$. **15.** £4200.

XIX. a. Pages 213, 214.

8. a^3+2b^3 is the greater. **12.** $x^3 >$ or $< x^2+x+2$, according as $x >$ or <2.

14. The greatest value of x is 1. **15.** 4; 8.

22. $4^4 . 5^5$; when $x=3$. **23.** 9, when $x=1$.

XIX. b. Pages 218, 219.

10. $\frac{3^3 . 5^5}{2^8} a^8$; $\sqrt{\frac{3}{5}} . \sqrt[3]{\frac{2}{5}}$.

Circle without a starting place. –

O - Line A = 10 x no. in circle.

no in circle = $\frac{\lfloor no}{no}$.

not -

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Binomial Theorem. —

From among $m + n + p$ things m things may be chosen in $\dfrac{|\underline{m+n+p}}{|\underline{n}\ |\underline{n+p}}$.

and in each case n things may be chosen from $n+p$ which remain in $\dfrac{|\underline{n+p}}{|\underline{n}\ |\underline{p}}$ ways.

Jointly $\dfrac{|\underline{m+n+p}}{|\underline{n}\,.\,|\underline{n+p}} \times \dfrac{|\underline{n+p}}{|\underline{n}\ |\underline{p}}$.

If definite destination then an order of chosing.

If just into parcels & not divid thro' by

3 persons $|\underline{3}$

tetra-hedoron

triangular pyramid

$$px^2+qx+r=0$$

$$x^2+\frac{qx}{p}=\frac{-r}{p}$$

$$x^2+\frac{qx}{p}+\frac{q^2}{4p^2}=\frac{-r}{p}+\frac{q^2}{4p^2}=\frac{-4pr+q^2}{4p^2}$$

$$x+\frac{q}{2p}=\frac{\pm\sqrt{q^2-4pr}}{2p}$$

$$x=\frac{-q\pm\sqrt{q^2-4pr}}{2p}$$

Roots are, —

1 real + unequal when b^2-4ac (>0) is positive

2 " + equal " $b^2-4ac=0$

3. Imaginary " $b^2-4ac<0$ ~~negative~~

4. Rational. b^2-4ac. is perfect sq.

11, 12, 13

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Feb 17th — Mathematics (6.7 8)

Feb 22nd — S. Geom. (6.7)

Feb 24 — — Trig. (6.7)

March 1 — (A Geom) (6.7)

7. $(2x - 3y)^{28}$ when $x = 9$ $y = 4$

Find greatest term

$$(2x - 3y)^{28} = 2x^{28}\left(1 - \frac{3y}{2x}\right)^{28}$$

$$t_r = \frac{28 \cdot 27 \cdot 26 \cdots\cdots 30-r}{1 \cdot 2 \cdot 3 \cdots\cdots r-1}\left(\frac{3y}{2x}\right)^{r-1}$$

$$t_{r+1} = \frac{28 \cdot 27 \cdot 26 \cdots\cdots 29-r}{1 \cdot 2 \cdot 3 \cdots\cdots r}\left(\frac{3y}{2x}\right)^{r}$$

$\therefore t_{r+1} > t_r$ so long as $\frac{29-r}{r} \cdot \frac{3y}{2x} > 1$

$$\therefore \frac{29-r}{r} \cdot \frac{12}{18} > 1$$

$$348 - 12r > 18r$$

$$r < 11\tfrac{18}{30}$$

$\therefore$ t_{12} is greatest term.

OF CANADA